High Dimensional Expanders Lecture 7

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Our goal is to describe a simple construction of bounded degree high dimensional expanders due to Kaufman and Oppenheim [KO18]. Our notes are based on the presentation of the construction given in [HS19].

1 Groups, subgroups and cosets

Before describing the construction, we give a quick overview of groups.

Definition (Group)**.** A group is a pair $(G, \cdot)$ where G is a non-empty set and $\cdot : G \times G \to G$ has the properties:

1. For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This is called associativity.
2. There exists and identity element $e \in G$ so that for all $a \in G$ it holds that $a \cdot e = e \cdot a = a$.
3. For every $a \in G$ there exists (a unique) $a^{-1} \in G$ so that $aa^{-1} = a^{-1}a = e$.

We sometimes just write ab instead of $a \cdot b$, and G instead of $(G, \cdot)$.

Examples:

1. The $(\mathbb{R}, \cdot)$ where $\cdot$ is the usual multiplication.
2. $(Z_n = \{0, 1, ..., n-1\}, +)$ where the operation is addition mod n.
3. Any vector space $(V, +)$ where the operation is just addition of vectors.
4. $(S_n, \circ)$ where $S_n = \{\sigma : [n] \to [n] \mid \sigma \text{ is invertible}\}$ and $\circ$ is composition of functions.
5. $(GL_n(\mathbb{F}_q), \cdot)$ where $GL_n(\mathbb{F}_q)$ is the set of invertible matrices of dimension $n \times n$, and $\cdot$ is matrix multiplication. We can

Remark 1.1. Notice that in the first three examples, the operation is *commutative*. that is $ab = ba$ for every $a, b \in G$. This doesn't hold in general for the last two examples. Groups with this property are called *abelian groups*.

1.1 Subgroups

Definition (Subgroup)**.** Let $(G, \cdot)$ be a group. A subgroup is a non-empty set $H \subseteq G$ so that $(H, \cdot)$ is a group. Equivalently, $e \in H$ and for every $a, b \in H$ $a \cdot b, a^{-1} \in H$. When H is a subgroups we write $H \leqslant G$ instead of $H \subseteq G$.

Obviously $\{e\}, G \leqslant G$. Other, non-trivial examples include:

1. $\mathbb{Q} \subseteq \mathbb{R}$ or $\mathbb{Z} \subseteq \mathbb{R}$.
2. $\{0, 2, 4, 6\} \subseteq Z_8$.
3. Any subspace $W \subseteq V$ (these aren't the only subgroups of a vector space, e.g. $\mathbb{Z}^2 \subseteq \mathbb{R}^2$).
4. $\{\sigma \in S_n \mid \sigma(n) = n\} \leqslant S_n$.
5. $SL_n(\mathbb{F}_q) \leqslant GL_n(\mathbb{F}_q)$ where $SL_n(\mathbb{F}_q) = \{A \in GL_n(\mathbb{F}_q) \mid det(A) = 1\}$.

Another way to construct a subgroup, is to say which elements "must" be in it. For a set $S \subseteq G$ let $\langle S \rangle \leqslant G$ be

$$\langle S \rangle = \left\{ g_1 g_2 ... g_m \,\middle|\, m \in \mathbb{N}, g_1, ..., g_m \in S \cup S^{-1} \right\}.$$

Another way to describe $\langle S \rangle = \bigcap_{S \subseteq H \leqslant G} H$ (it is an easy exercise showing this is well defined and that both definitions are equivalent).

1.2 Cosets

Let $H \subseteq G$ and $g \in G$. The set $gH = \{gh \mid h \in H\}$ is a coset (if and only if $g \in g'H$). Note that $gH = g'H$ if and only if $g^{-1}g' \in H$. Otherwise, $gH \cap g'H = \emptyset$. In other words, the cosets of H are a partition of G that comes from the relation is $g\tilde{g}' \Leftrightarrow g^{-1}g' \in H$. The *index* of H is the number of cosets, which is $|G|/|H|$. It is denoted $[G : H]$.

For example, if $H = \{(t, 0, 0) \mid t \in \mathbb{F}_q\} \subseteq \mathbb{F}_q^3 = G$. then its cosets vH (or in the more common additive notation $v + H$) are the affine lines $\{(b_1 + t, b_2, b_3) \mid t \in \mathbb{F}_q\}$ for every $(b_1, b_2, b_3) \in G$.

2 Coset complex and examples

In this section we present the construction for triangle complexes. The general construction is described in [KO18].

Let G be a group and let $K_1, K_2, K_3, ..., K_d$ be three subgroups. The of our construction are $X(0) = A_1 \cup A_2 \cup A_3 ... \cup A_d$ where $A_i = \{gK_i \mid g \in G\}$. The $(d-1)$-faces are

$$X(d-1) = \left\{ \{g_1K_1, g_2K_2, ..., g_dK_d\} \,\middle|\, \bigcup_{i=1}^{d} g_iK_i \neq \emptyset \right\}.$$

For the rest of this talk, $d \in \{2, 3\}$.

Let us describe a couple of examples.

Example 3. Let $G = \mathbb{F}_q^3$. Let $K_x = \{(0, y, z) \mid y, z \in \mathbb{F}_2\}$, $K_y = \{(x, 0, z) \mid x, z \in \mathbb{F}_2\}$ and $K_z = \{(x, y, 0) \mid x, y \in \mathbb{F}_2\}$.

- $v_1 K_x v_2 K_x$ if and only if their first coordinate is the same. Similarly for K_y, K_z.
- Hence there are q-cosets for every subgroup.
- Every possible triangle between the three parts (A_i, A_j, A_k) participates in X. Since every three of the affine plains above intersect in a point.
- Thus, this example yields the complete 3-complex with faces of size q.

Example 4. Recall that $G = S_4$ are all permutations on four elements. Let $K_i = \{\sigma \in S_4 \mid \sigma(i) = i\}$. We take as our subgroups $K_1, K_2, K_3 \subseteq S_4$.

- One can verify that the coset of τ so that $\tau(i) = j$ is $\tau K_i = \{\sigma \in S_4 \mid \sigma(i) = j\}$.
- Denote such the cosets of K_i by $K_{i \to j} = \{\sigma \in S_4 \mid \sigma(i) = j\}$.
- The triangles are all $\{K_{1\to i}, K_{2\to j}, K_{3\to k}\}$ so that i, j, k are distinct.
- In particular, a link of a face is a 6-cycle.

4.1 Basic Properties

Claim 4.1. The triangle $\{g_1K_1, g_2K_2, g_3K_3\} \in X$ if and only if $g_1K_1 \cap g_2K_2 \cap g_3K_3 \neq \emptyset$.

Proof. If $\{g_1K_1, g_2K_2, g_3K_3\} \in X$ then there is a g so that $gK_1 = g_1K_1, gK_2 = g_2K_2, gK_3 = g_3K_3$. In particular g is in the intersection and it is not empty.

On the other hand, take some $\{g_1K_1, g_2K_2, g_3K_3\}$ with a non-empty intersection, and take some g in the intersection. In particular it holds that $gK_i = g_iK_i$ so this face is equal to $\{gK_1, gK_2, gK_3\}$. ◻

As a corollary to this claim we see that the intermediate faces consist of cosets with non-trivial intersection, that is $X(1) = \left\{\{g_1K_i, g_2K_j\} \;\middle|\; g_1K_i \cap g_2K_j \neq \emptyset\right\}$.

Next we will investigate when this complex is connected.

Lemma 4.2. *The* 1*-skeleton of* X *is connected if and only if* $\langle K_1 \cup K_2 \cup K_3 \rangle = G$*. That is, if for every* $g \in G$ *there exists* $m \in \mathbb{N}$ *and* $\{g_i \in K_{j_i}\}_{i=1}^m$ *so that* $g = g_1 \cdot g_2 \cdot \ldots \cdot g_m$*.*

To show this we need the following useful claim.

Claim 4.3. $g_1K_1 \sim g_2K_2$ if and only if $g_1^{-1}g_2 \in K_1K_2$ (i.e. there exists $h_1 \in K_1, h_2 \in K_2$ so that $g_2^{-1}g_1 = h_1h_2$).

Proof of Claim 4.3. On the one hand, if $g_1K_i \sim g_2K_2$ then there is some g so that $gK_i = g_iK_i$. In particular, $g^{-1}g_1 \in K_1, g^{-1}g_2 \in K_2$ and thus $g_1^{-1}gg^{-1}g_2 \in K_1K_2$.

On the other hand, if $g_1^{-1}g_2 = k_1k_2$ then $g_1k_1 = g_2k_2 \in g_1K_1 \cap g_2K_2$. ◻

Proof of Lemma 4.2. We show that for every g there is some $i \in \{1, 2, 3\}$ we can get from K_1 to gK_i (and then we can get to any gK_j by one more edge). Assume that $g = g_1g_2 \ldots g_m$ where $g_j \in K_{i_j}$. Our path will be $P = (K_1, g_1K_{i_1}, (g_1g_2)K_{i_2}, \ldots, gK_{i_m}$. Note that for every j, the edge $(g_1g_2 \ldots g_j)K_{i_j}, (g_1g_2 \ldots g_jg_{j+1})K_{i_{j+1}}$ exists. Indeed, note that $(g_1g_2 \ldots g_j)^{-1}(g_1g_2 \ldots g_jg_{j+1}) \in K_{i_{j+1}} \subseteq K_{i_j}K_{i_{j+1}}$ hence by Claim 4.3 the edge appears in X.

Assume that the graph is connected, and we show that G is generated by the K_i's. Let $g \in G$ and consider a path from K_1 to gK_1, $P = (K_1, g_1K_{i_2}, g_3K_{i_3}, ..., g_mK_1 = gK_1)$. Note that $g_j^{-1}g_{j+1} \in K_{i_j}K_{i_{j+1}} \subseteq \langle K_1, K_2, K_3\rangle$. Hence

$$g = g_m = g_{m-1}(g_{m-1}^{-1}g) = g_{m-2}(g_{m-2}^{-1}g_{m-1})(g_{m-1}^{-1}g) = ...$$

$$= g_1(g_1^{-1}g_2)...(g_{m-1}^{-1}g).$$

□

Next we show that this complex is very symmetric.

Claim 4.4. For every $g \in G$ there is an automorphism of simplicial complexes $\psi : X(0) \to X(0), \psi(hK_j) = (g^{-1}h)K_j$ that sends gK_i to K_i. In particular, the links of all gK_i are isomorphic to the link of K_i (the subgroup itself).

Proof. The non-trivial part of proving this is to show that ψ is well defined. That is, that if $h_1K_j = h_2K_j$ then $g^{-1}h_1K_j = g^{-1}h_2K_j$. But this is true since $(g^{-1}h_1)^{-1}g^{-1}h_2 = h_1^{-1}h_2$. The rest is follows easily from the construction. □

As a corollary we get that the links of gK_i are isomorphic to K_i.

Claim 4.5. The link of $v = gK_i$ is (isomorphic to) the bipartite graphs between cosets of $K_i \cap K_j$ and $K_i \cap K_\ell$ where two vertices are connected if their intersection isn't empty. That is $X_{gK_i} \cong X(k_i; K_i \cap K_j, K_i \cap K_\ell)$.

Proof. By Claim 4.4 we can prove this without loss of generality on $v = K_1$. The link of K_1 consists of vertices $X_{K_1}(0) = \{gK_2, gK_3 \mid\ g \in K_1\}$ and edges $X_{K_1}(1) = \{\{gK_2, gK_3\} \mid\ g \in K_1\}$. Define an isomorphism from $\phi : X_v \to X(K_1; K_1 \cap K_2, K_1 \cap K_3)$ defined by $\phi(gK_i) = g(K_1 \cap K_i)$ where $g \in K_1$. This is well defined in the link, since $gK_i = g'K_i$ and $g, g' \in K_1$ imply that $g'^{-1}g \in K_1 \cap K_i$ which implies that $g(K_1 \cap K_i) = g'(K_1 \cap K_i)$.

Checking that this is a bijection and that it preserves edges is left to the reader. □

5 Concrete complexes

Let q be a prime power. Let $R = \mathbb{F}_q[t]/\langle t^m\rangle$ (i.e. all polynomials of degree $\leqslant m-1$ where our multiplication is done modulo $t^m = 0$).

The group G we will use is $G = SL_3(R) = \{A \in M_3(R) \mid\ det(A) = 1\}$. Our three subgroups will be

$$K_1 = \left\{ \begin{pmatrix} 1 & \ell_1(x) & Q(x) \\ 0 & 1 & \ell_2(x) \\ 0 & 0 & 1 \end{pmatrix} \in M_3(R) \,\middle|\, deg(\ell_1(x)), \deg(\ell_2(x)) \leqslant 1, deg(Q(x)) \leqslant 2 \right\},$$
$$K_2 = \left\{ \begin{pmatrix} 1 & 0 & 0) \\ Q(x) & 1 & \ell_1(x) \\ \ell_2(x) & 0 & 1 \end{pmatrix} \in M_3(R) \,\middle|\, deg(\ell_1(x)), \deg(\ell_2(x)) \leqslant 1, deg(Q(x)) \leqslant 2 \right\}, \quad (5.1)$$
$$K_3 = \left\{ \begin{pmatrix} 1 & \ell_1(x) & 0 \\ 0 & 1 & 0 \\ \ell_2(x) & Q(x) & 1 \end{pmatrix} \in M_3(R) \,\middle|\, deg(\ell_1(x)), \deg(\ell_2(x)) \leqslant 1, deg(Q(x)) \leqslant 2 \right\},$$

These groups may look arbitrary, and indeed the reason they yield high dimensional expanders is because some representation theoretic properties they have, which are not apparent in first sight. Still there is an elementary description of their links.

5.1 Number of vertices and connectivity

Observation 5.1. The size of every K_i is q^7 and $\lim_{m\to\infty}|G| = \infty$. Thus by taking $m \to \infty$ we get a family of high dimensional expanders that grow to infinity.

Moreover, these three groups generate G. A proof for this fact could be found in [KO18, Section 3] (see also [HS19] for a simpler proof that the size of $\langle K_1, K_2, K_3\rangle$ goes to infinity). By claim Claim 4.5, the link structure doesn't depend on G at all; it is always (isomorphic to) $X(K_i; K_i \cap K_j, K_i \cap K_\ell)$. Hence these complexes are a growing family of bounded degree connected simplicial complexes.

5.2 Structure of the links

In this section we describe the links of X. That is, we understand the structure of $X(K_i; K_i \cap K_j, K_i \cap K_\ell)$.

We will see the structure of links of type K_1. The other links look the same. Let $H_2 = K_1 \cap K_2, H_3 = K_1 \cap K_3$ and let $Y = X(K_1, H_2, H_3)$.

Claim 5.2. Let

$$M_2(\ell(x), Q(x)) = \begin{pmatrix} 1 & \ell(x) & Q(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_3(\ell(x), Q(x)) = \begin{pmatrix} 1 & 0 & Q(x) \\ 0 & 1 & \ell(x) \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

1. Every coset H_2 has a unique representative $M_2(\ell(x), Q(x))$.
2. Every coset H_3 has a unique representative $M_3(\ell(x), Q(x))$.
3. Every coset $M_2(\ell(x), Q(x))H_2 \sim M_3(\ell'(x), Q'(x))H_3$ if and only if $Q'(x) - Q(x) = \ell(x)\ell'(x)$.

Proof. For the first item we notice that $|K_1| = q^7$ and $||H_i|| = q^2$, so there are q^5 different cosets. Hence to show the first item it is enough to show that $M_2(\ell(x), Q(x))H_2 \neq M_2(\ell'(x), Q'(x))H_2$ whenever $M_2(\ell(x), Q(x)) \neq M_2(\ell'(x), Q'(x))$, since this implies that the $M_2(\ell(x), Q(x))$ define all possible q^5 cosets. Indeed, by direct calculation it holds that $M_2(\ell(x), Q(x))H_2 = M_2(\ell'(x), Q'(x))H_2$ if and only if $M_2(\ell(x), Q(x))^{-1} \cdot M_2(\ell'(x), Q'(x)) \in H_2$. By a direct calculation this is

$$M_2(\ell(x), Q(x))^{-1} \cdot M_2(\ell'(x), Q'(x)) = \begin{pmatrix} 1 & \ell'(x) - \ell(x) & Q'(x) - Q(x) \\ 0 & 1 & \ell(x) \\ 0 & 0 & 1 \end{pmatrix}.$$

This is in H_2 if and only if the non-diagonal entries in the first row are zero, which is if and only if $\ell = \ell', Q = Q'$ and the first item follows.

The proof of the second item is similar to the first item.

As for the third item, we use Claim 4.3. It is easy to check that

$$(M_2(\ell(x), Q(x))^{-1}M_3(\ell'(x), Q'(x)) = \begin{pmatrix} 1 & -\ell(x) & Q'(x) - Q(x) - \ell(x)\ell'(x) \\ 0 & 1 & \ell'(x) \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, one may verify that $H_2H_3 = \{h_2h_3 \mid h_2 \in H_2, h_3 \in H_3\}$ is equal to

$$\left\{ \begin{pmatrix} 1 & \ell_1(x) & 0 \\ 0 & 1 & \ell_2(x) \\ 0 & 0 & 1 \end{pmatrix} \middle| \; deg(\ell_1(x)), deg(\ell_2(x)) \leqslant 1 \right\}.$$

Thus by Claim 4.3, $M_2(\ell(x), Q(x))H_2 \sim M_3(\ell'(x), Q'(x))H_3$ if and only if the top right entry is 0) □

Hence the following bipartite graph $G = (L, R, E)$ is an equivalent description to Y:

$$L = R = \{(\ell(x), Q(x)) \mid \; deg(\ell(x)) \leqslant 1, deg(Q(x)) \leqslant 2\},$$

$$E = \left\{(Q, \ell(x)), (Q'(x), \ell'(x)) \,\middle|\; Q'(x) - Q(x) = \ell(x)\ell'(x)\right\}.$$

These graphs are expanders.

Claim 5.3. The links of X are $\frac{1}{\sqrt{q}}$-one-sided spectral expanders.

We do not prove this here. For a representation theoretic proof of this see [KO18], for a more elementary proof see [HS19]. A third proof that is very simple, provided you feel comfortable with Cayley graphs and the Schwartz-Zippel lemma, is done by [OP22].

References

[KO18] Tali Kaufman and Izhar Oppenheim. "Construction of New Local Spectral High Dimensional Expanders." In: *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2018. Los Angeles, CA, USA: Association for Computing Machinery, 2018, pp. 773–786. ISBN: 9781450355599. DOI: 10.1145/3188745.3188782. URL: https://doi.org/10.1145/3188745.3188782.

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