On the Hardness of Approximating Label Cover

Irit Dinur* Shmuel Safra*

November 1, 2002

Abstract

The Label-Cover problem, defined in [ABSS93], serves as a starting point for numerous hardness of approximation reductions. It is one of six ‘canonical’ approximation problems in the survey of Arora and Lund [AL97]. In this paper we present a direct combinatorial reduction from low error-probability PCP [DFK+99] to Label-Cover showing it NP-hard to approximate to within $2^{\log n^{1-\varepsilon}}$. This improves upon the best previous hardness-of-approximation results known for this problem.

We also consider the Minimum-Monotone-Satisfying-Assignment (MMSA) problem of finding a satisfying assignment to a monotone formula with least number of 1s, [ABMP98]. We define a hierarchy of approximation problems obtained by restricting the number of alternations the monotone formula. This hierarchy turns out to be equivalent to an AND/OR scheduling hierarchy suggested in [GM97]. We show some hardness results for certain levels in this hierarchy, and place Label-Cover between levels 3 and 4. This partially answers an open problem from [GM97] regarding the precise complexity of each level in the hierarchy, and the place of Label-Cover in it.

1 Introduction

The Label-Cover problem is a combinatorial graph labelling problem defined as follows. The input is a bipartite graph $G=(U,V,E)$, two sets of labels, $B_1$ for $U$ and $B_2$ for $V$, and for each edge $(u,v) \in E$, a relation $\Pi_{u,v} \subseteq B_1 \times B_2$ consisting of admissible pairs of labels for that edge. A labelling $(f_1, f_2)$ is a pair of functions $f_1: U \to 2^{B_1}, f_2: V \to 2^{B_2}$ assigning a subset of labels to each vertex. A labelling covers an edge $(u,v)$ if for every label $a_2 \in f_2(v)$ there is a label $a_1 \in f_1(u)$ such that $(a_1, a_2) \in \Pi_{u,v}$. The goal is to find a labelling that covers all edges such that the $l_p$ norm of the vector $(|f_1(u_1)|, |f_1(u_2)|, \ldots, |f_1(u_m)|) \in \mathbb{Z}^{|U|}$ is minimized.

This problem was shown (implicitly in [LY94] and more formally in [ABSS93]) quasi-NP-hard to approximate to within a factor of $2^{\log^{1-\varepsilon} n}$ for any constant $\delta > 0$ by showing a specific two-prover one-round interactive proof protocol, which reduces to Label-Cover.

We prove that Label-Cover is NP-hard to approximate to within $2^{\log^{1-\varepsilon} n}$ where $\delta = \log \log^{-c} n$ for any $c < 1/2$. This improves the best previously known results achieving NP-hardness rather than quasi-NP-hardness, and obtaining a larger factor for which hardness of-approximation is proven. Our result also immediately strengthens the results of [GM97, ABMP98] and shows that the following problems are NP-hard to approximate to within a factor of $2^{\log^{1-1/\log \log^{-c} n}}$ for any $c < 1/2$: MMSA, Minimum-Length-Frege-Proof, Minimum-Length-Resolution-Refutation, AND/OR scheduling, Linear-Remove-Part, Remove-Part, Separate-Pair, Full-Disassembly, Remove-Set, and Separate-Set.

* School of Mathematical Sciences, Tel Aviv University, ISRAEL
Remark. In [ABSS93], LABEL-COVER was reduced to the CLOSEST-VECTORS problem, the NEAREST CODEWORD problem, MAX-SATISFY, MIN-UNSATISFY, learning half-spaces in the presence of errors, and a number of other problems. Unfortunately, their reduction, is not from general LABEL-COVER, but rather relies on a special additional property of the LABEL-COVER instance that they construct. Namely that the relations associated with each edge are partial functions: every label for $u$ can be covered by at most one label for $v$. This property is inherently missing in our reduction, and indeed hardness results for the aforementioned problems seem to require more work than is in our direct reduction.

A Formula-Depth Hierarchy

We also consider a related problem called MINIMUM-MONOTONE-SATISFYING-ASSIGNMENT (MMSA) that was defined in [ABMP98], and shown there to be as hard as LABEL-COVER. Given a monotone formula $\varphi$ the problem is to find a satisfying assignment for $\varphi$ with a minimum number of 1's. This problem was considered in [ABMP98] since it reduces to the problem of finding the length of a propositional proof, a problem of considerable interest in proof theory. Although our LABEL-COVER result strengthens the hardness for MMSA, we note that [Uma99] subsequently obtained an even better $n^{1-\varepsilon}$ hardness result for this problem without going through a reduction from LABEL-COVER.

We show that the MMSA problem can be viewed as a generalization of the LABEL-COVER problem. We examine a hierarchy of approximation problems formed by restricting the depth of the monotone formula in the MMSA problem. This hierarchy is equivalent to a hierarchy of AND/OR scheduling pointed out in [GM97]. A monotone formula is said to be of depth $i$ if it has $i-1$ alternations between AND and OR. A depth-$i$ formula is called $\Pi_i$ (or $\Sigma_i$) if the first level of alternation is an AND (or OR). It is easy to see that the complexity of MMSA restricted to $\Sigma_{i+1}$ formulas is equivalent the complexity of MMSA restricted to $\Pi_i$ formulas, denoted $\text{MMSA}_i$.

Each $\text{MMSA}_i$ is at least as hard to approximate as $\text{MMSA}_{i-1}$. $\text{MMSA}_1$ is trivially solvable in polynomial time. $\text{MMSA}_2$ is already quite harder, and actually a simple approximation-preserving reduction from SET-COVER to $\text{MMSA}_2$ was shown in [ABMP98], implying that $\text{MMSA}_2$ is NP-hard to approximate to within logarithmic factors [RS97]. In fact, the two problems can be easily shown to be equivalent, thus the same greedy algorithm for SET-COVER [Joh74, Lov75] approximates $\text{MMSA}_2$ to within a factor of $\ln n$. We know of no previous hardness result for $\text{MMSA}_3$. A reduction from LABEL-COVER to $\text{MMSA}_4$ was shown independently in [ABMP98] and [GM97].

We show how to translate $\text{MMSA}_3$ to LABEL-COVER, altogether placing LABEL-COVER somewhere between levels 3 and 4 in this hierarchy. This partially answers an open question from [GM97] of whether or not LABEL-COVER is equivalent to level 4 in the hierarchy. Furthermore, we examine the (previously unknown) hardness of $\text{MMSA}_3$ and via a reduction from PCP to $\text{MMSA}_3$ show that it is NP hard to approximate to within the above large factors. This immediately follows through for $\text{MMSA}_i$ (for every $i \geq 3$) and for LABEL-COVER. Our reductions all involve a polynomial sized blow-up, thus the hardness-of-approximation ratios are polynomially related. For the asymptotic approximation ratios discussed here, this polynomial blow-up is irrelevant.

If we denote the relation reducible with a polynomially related approximation-ratio by $\ll$ we can write:

$$\text{PCP} \ll \text{MMSA}_3 \ll \text{LABEL-COVER} \ll \text{MMSA}_4 \ll \ldots \ll \text{MMSA}_i$$
We summarize the above in the following table:

<table>
<thead>
<tr>
<th>Formula Depth</th>
<th>Approximation Algorithm</th>
<th>NP-Hardness Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMSA_1</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>MMSA_2</td>
<td>ln n</td>
<td>(\Omega(\log n))</td>
</tr>
<tr>
<td>MMSA_{&gt;3}</td>
<td>n</td>
<td>(2^{\log^{1-o(1)} n})</td>
</tr>
</tbody>
</table>

**Technique**

We show a direct reduction to \textsc{Label-Cover} from low error-probability PCP with parameters \(D\) and \(\varepsilon\). Namely, we begin with a gap-SAT instance consisting of Boolean functions. These Boolean functions each depend on \(D\) variables, and the variables range over \(\{1 \ldots 1/\varepsilon\}\). The PCP theorem states that it is NP-hard to distinguish between the ‘yes’ case where the whole system is satisfiable, and the ‘no’ case where every assignment satisfies no more than an \(\varepsilon\) fraction of the local-tests. The focus of [DFK+99] was on \(D = O(1)\), and thus only an error-probability of \(\varepsilon = 2^{-\log^{1-\delta} n}\) for any constant \(\delta > 0\) was claimed. This alone strengthens the hardness of \textsc{Label-Cover} from quasi-NP-hardness to NP-hardness, but with the same hardness-factor as before. For our purposes however, the best result is obtained by choosing \(D = \log \log^c n\) for any \(c < 1/2\) and \(\varepsilon = 2^{-\log^{1-o(1)} n}\). These parameters give the result claimed above. Notice that our direct reduction immediately implies that a stronger PCP characterization of NP – e.g. one with a polynomially-small error-probability and constant depend as conjectured in [BGLR93] – would immediately give NP-hardness for approximating \textsc{Label-Cover} to within \(n^c\) for some constant \(c > 0\).

**Structure of the Paper**

Our main result for \textsc{Label-Cover} is proven in section 2. The hardness result for MMSA_3 is proven in section 3, via a reduction from PCP. We then show, in section 4 a reduction from MMSA_3 to \textsc{Label-Cover} thus placing \textsc{Label-Cover} between levels 3 and 4 in the ‘MMSA’ hierarchy. This also re-establishes the hardness result for \textsc{Label-Cover} already shown in section 2.

## 2 Label Cover

The \textsc{Label-Cover} problem is defined as follows.

**Definition 1 (Label-Cover (LC_p))** The input to the label-cover problem is a bipartite graph \(G = (U,V,E)\), two sets of labels, \(B_1\) for \(U\) and \(B_2\) for \(V\), and for each edge \((u,v) \in E\), a relation \(\Pi_{u,v} \subseteq B_1 \times B_2\) consisting of admissible pairs of labels for that edge. A labelling \((f_1,f_2)\) is a pair of functions \(f_1 : U \rightarrow \{0,1\}^{B_1}\), \(f_2 : V \rightarrow \{0,1\}^{B_2}\) assigning a subset of labels to each vertex. The \(l_p\)-cost of the labelling is the \(l_p\) norm of the vector \((|f_1(u_1)|, |f_1(u_2)|, \ldots, |f_1(u_m)|) \in \mathbb{Z}_+^{B_1}\). A labelling covers an edge \((u,v)\) if for every label \(a_2 \in f_2(v)\) there is a label \(a_1 \in f_1(u)\) such that \((a_1,a_2) \in \Pi_{u,v}\). A total-cover of \(G\) is a labelling that covers every edge. The problem \(LC_p\) is to find a total-cover with minimal \(l_p\)-cost \((1 \leq p \leq \infty)\).

In this section we show a direct reduction from PCP to \textsc{Label-Cover} with \(l_p\) norm, \(1 \leq p \leq \infty\), such that the approximation factor is preserved.
Let us denote $g_c(n) \overset{\text{def}}{=} 2^{\log^{1-1/\log\log^c n} n}$. Our reduction will imply that LABEL-COVER is NP-hard to approximate to within factor $g_c(n)$ for any $c < 1/2$. Our starting point is the PCP theorem from [DFK+99],

**Theorem 1 (PCP Theorem [DFK+99])** Let $c < 1/2$ be arbitrary and let $D \leq \log \log^c n$. Let $\Psi = \{\psi_1, ..., \psi_n\}$ be a system of Boolean functions over variables $X = \{x_1, ..., x_{n'}\}$ such that each Boolean function depends on $D$ variables, and each variable ranges over $\mathcal{F}$ where $|\mathcal{F}| = O(2^{(\log n)^{1-1/O(D)}})$. It is NP-hard to distinguish between the following two cases:

**Yes:** There is an assignment to the variables such that all $\psi_1, ..., \psi_n$ are satisfied.

**No:** No assignment can satisfy more than $\frac{1}{|\mathcal{F}|}$ fraction of the $\psi_i$'s.

In this section we prove LABEL-COVER to be NP-hard to approximate to within a factor of $g$, where $g = g_c(n)$ is fixed for some arbitrary $c < 1/2$.

**Theorem 2** For any $c < \frac{1}{2}$, and any $1 \leq p \leq \infty$, LABEL-COVER$_p$ is NP-hard to approximate to within a factor of $g = g_c(n)$.

**Proof:** The proof follows by reduction from PCP. Choose some $c < c' < 1/2$, let $\mathcal{F}$ be such that $|\mathcal{F}| = O(g_{c'}(n))$, and let $\Psi = \{\psi_1, ..., \psi_n\}$ be a PCP instance as in the above theorem. For a test $\psi \in \Psi$ and a variable $x \in X$, we write $x \in \psi$ when $\psi$ depends on $x$, and denote $\Psi_x \overset{\text{def}}{=} \{\psi \in \Psi \mid x \in \psi\}$.

We construct from $\Psi$ a bipartite graph $G = (U, V, E)$ with $U \overset{\text{def}}{=} \{u_1, ..., u_{nD}\}$ consisting of a vertex for every appearance of a variable in $\Psi$ and $V \overset{\text{def}}{=} [n]$ consisting of a vertex for every test $\psi \in \Psi$. We denote $U(x) \subset U$ the set of vertices corresponding to the variable $x$. A vertex $j \in V$ is connected to all appearances of the variables in $\psi_j$. Formally,

$$E \overset{\text{def}}{=} \{(u, j) \mid u \in U(x) \text{ and } x \in \psi_j\}$$

We set $B_1 \overset{\text{def}}{=} \mathcal{F}$ and $B_2 \overset{\text{def}}{=} \mathcal{F}^D$. For an edge $(u, j) \in E$, assume $u \in U(x)$ and $x$ is the $i$th variable in $\psi_j$, and define

$$\Pi_{u, j} = \{(a_i, (a_1, ..., a_D)) \mid \psi_j(a_1, ..., a_D) = \text{True}\}$$

**Proposition 1 (Completeness)** If there is a satisfying assignment for $\Psi$, then there is a total-cover for $G$ with $l_\infty$-cost $1$, and $l_1$-cost $n \cdot D$.

**Proof:** Let $\mathcal{A} : X \rightarrow \mathcal{F}$ be an assignment satisfying all of $\Psi$. Define for each $u \in U(x)$, $f_1(u) \overset{\text{def}}{=} \{\mathcal{A}(x)\}$ and $f_2(v_j) \overset{\text{def}}{=} \{(\mathcal{A}(x_{i_1}), ..., \mathcal{A}(x_{i_D})) \mid \psi_j$ depends on $x_{i_1}, ..., x_{i_D}\}$ (these are both singleton sets). This is a total-cover of $l_\infty$ cost $1$ and $l_1$-cost $n \cdot D$. $\blacksquare$

We next show that if $\Psi$ is a ‘no’ instance, then any label-cover has $l_\infty$ cost more than $g$. This is formulated in a contrapositive manner as follows.

**Proposition 2 (Soundness$_\infty$)** If there is a total-cover for $G$ with $l_\infty$-cost $g$, then there is an assignment $\mathcal{A}$ satisfying $g^{-D} > \frac{1}{|\mathcal{F}|}$ fraction of $\Psi$ (and $\Psi$ is not a ‘no’ instance).
Proof: Let \((f_1, f_2)\) be a labelling for \(G\) that is a total-cover with \(l_{\infty}\)-cost \(g\), i.e.

\[
\max_i(|f_i(v_i)|) = g.
\]

We define a random assignment \(A\) for the variables \(X\) by choosing for every variable \(x_i\) a value uniformly at random from \(f_1(u)\) where \(u \in U(x_i)\) is an arbitrary vertex in \(U(x_i)\). Each label \(r \in f_2(v_i)\) corresponds to an assignment that satisfies \(\psi_i\) and such that \(r|_{x_i} \in f_1(u)\) for every vertex \(u \in U(x_i)\) and variable \(x_i\) appearing in \(\psi_i\). Thus, a test \(\psi_j\) is satisfied with probability \(|f_2(v_j)|/g^D \geq g^{-D}\), so the expected number of tests satisfied by \(A\) is also \(\geq g^{-D}\). There must be an assignment that attains the expectation, and satisfies at least \(g^{-D}\) fraction of the tests in \(\Psi\).

Note that for the \(g = g_c(n)\) chosen above \(g^{-D} > \frac{1}{|\mathcal{F}|}\) because \(|\mathcal{F}| = O(g_c(n))\) for \(c' > c\), thus \(\Psi\) is not a ‘no’ instance. \(\blacksquare\)

We next show that if \(\Psi\) is a ‘no’ instance, then any label-cover has \(l_1\) cost more than \(q\). This again, is formulated in a contrapositive manner as follows.

**Proposition 3 (Soundness)\(\) If there is a total-cover for \(G\) with \(l_1\)-cost \(g \cdot nD\), then there is an assignment \(A\) satisfying \(\geq \frac{1}{2} \cdot \frac{1}{(2D-g)^D} > \frac{1}{|\mathcal{F}|}\) fraction of \(\Psi\).

Proof: Let \((f_1, f_2)\) be a total-cover with \(l_1\) cost \(g \cdot nD\). For every variable \(x\), define \(A(x) \overset{def}{=} \bigcap_{u \in U(x)} f_1(u) \subseteq \mathcal{F}\) (this set is non-empty since \((f_1, f_2)\) is a total cover). Recall \(\Psi_x \subseteq \Psi\) denoted the set of tests that depend on \(x\). If \(u \in U(x)\) then 

\[
\sum_{x_i} |\Psi_x| \cdot |A(x_i)| \leq \sum_{u \in U} |f_1(u)| = g \cdot nD. \quad (\star)
\]

Consider the random procedure of choosing a test \(\psi \in \mathcal{R} \Psi\) uniformly at random and then choosing a variable \(x \in \mathcal{R} \Psi\) uniformly at random. The probability of choosing \(x\) is \(\frac{|\Psi_x|}{nD}\). Equation \(\star\) is equivalent to \(E(|A(x)|) \leq g\) where \(E(|A(x)|)\) denotes the expectation of \(|A(x)|\) for \(x\) is chosen by the above random procedure.

We call a variable \(x\) for which \(|A(x)| > 2D \cdot g\), a bad variable. By the Markov inequality

\[
\Pr_x [|f_1(x)| > 2D \cdot g \cdot E(|A(x)|)] < \frac{1}{2D}
\]

which means that the probability of hitting a bad variable is less than \(\frac{1}{2D}\).

\[
\frac{1}{2D} \geq \max_{\psi \in \Psi_x, x \in \psi} \Pr \left[ x \text{ is bad} \right] = \max_{\psi \in \mathcal{R} \Psi, x \in \psi} \Pr \left[ \psi \text{ contains a bad variable} \right] \cdot \Pr \left[ x \text{ is bad} \mid \psi \text{ contains a bad variable} \right]
\]

\[
\geq \max_{\psi \in \mathcal{R} \Psi, x \in \psi} \Pr \left[ \psi \text{ contains a bad variable} \right] \cdot \frac{1}{D}
\]

Multiplying by \(D\), we deduce that at least half of the tests \(\psi \in \mathcal{R} \Psi\) contain no bad variable. Next, define a random assignment \(A\) for \(\Psi\) by choosing, for every variable \(x\), a random value \(a \in A(x)\), \(A(x) \overset{def}{=} a\). For a test \(\psi_i\) and a value \(r \in f_2(v_i)\), the probability that each variable \(x \in \psi_i\) was assigned \(a = r|_{x}\) is \(\Pi_{x \in \psi_i} |A(x)|\) (recall that \(r\) satisfies \(\psi_i\) so this is a lower bound on
the probability that $\psi_1$ is satisfied by $A$). For tests that contain no bad variable, this probability is $\geq \frac{1}{(2D-q)^p}$. Hence the expected fraction of tests (of those containing no bad variable) that are satisfied by $A$ is $\geq \frac{1}{(2D-q)^p}$. Thus, there exists an assignment $A$ that attains this expectation, i.e. that satisfies $\geq \frac{1}{(2D-q)^p}$ fraction of the tests that contain no bad variables. Thus $A$ satisfies a $\geq \frac{1}{(2D-q)^p}$ fraction of all of the tests.

Note that for the above chosen $g = g_c(n)$, $\frac{1}{(2D-q)^p} \geq 1/|\mathcal{F}|$, thus $\Psi$ is not a ‘no’ instance.

Propositions 1 and 3 imply that distinguishing between the case where there is a total-cover for $G$ whose $l_1$ cost is $nD$ or $g \cdot nD$ would enable distinguishing between ‘yes’ and ‘no’ PCP instances, hence it is NP-hard. Similarly, Propositions 1 and 2 imply the same about distinguishing between the case where there is a total-cover for $G$ whose $l_\infty$ cost is 1 or $g$. The above can easily be generalized for any $l_p$ norm, $1 \leq p \leq \infty$.

3 Reducing PCP to MMSA$_3$

The Minimum-Monotone-Satisfying-Assignment (MMSA) problem is defined as follows.

**Definition 2 (MMSA)** Given a monotone formula $\varphi(x_1, ..., x_k)$ over the basis $\{\wedge, \vee\}$, find a satisfying assignment $A : \{x_1, ..., x_k\} \rightarrow \{0, 1\}$, (i.e. such that $\varphi(A(x_1), ..., A(x_k)) = \text{True}$), minimizing the weight $\sum_{x \in A} A(x_i)$.

MMSA$_3$ is the restriction of MMSA to formulas of depth-$i$. For example, MMSA$_3$ is the problem of finding a minimal-weight assignment for a formula written as an AND of ORs of ANDs.

In this section we show a direct reduction from PCP to MMSA$_3$, that preserves the approximation factor.

**Theorem 3** For any $c < \frac{1}{2}$, it is NP-hard to approximate MMSA$_3$ to within $g_c(n) \equiv 2^{\log^{1-1/\log\log^c n} n}$.

**Proof:** Again, our starting point is the low error-probability PCP theorem, Theorem 1. Fix $g = g_c(n)$, and fix $c < c' < 1/2$ arbitrarily. Take $\mathcal{F}$ to be such that $|\mathcal{F}| = O(g_c(n))$, and $D = O(\log \log^c n)$. Let $\Psi$ be a PCP instance as in Theorem 1. For a fixed $\psi \in \Psi$, we denote the set of satisfying assignments for it $R_\psi \subseteq \mathcal{F}$. For an assignment $r \in R_\psi$ and a variable $x \in \psi$ we write $r|_x \in \mathcal{F}$ to denote the restriction of $r$ to $x$.

We construct the monotone formula $\Phi$ over the following set of literals

$$T \equiv \bigcup_{x \in \lambda} \{T[x, \psi, a] \mid \psi \in \Psi_x, a \in \mathcal{F}\}.$$

This set has cardinality $nD \cdot |\mathcal{F}|$. The pair of variable $x$ and assignment $a$ for it will be represented by the conjunction $L[x, a] \equiv \bigwedge_{\psi \in \Psi_x} T[x, \psi, a]$ that can be read as “$a$ is assigned to $x$”. We define the formula $\Phi(T)$ by

$$\Phi(T) \equiv \bigwedge_{\psi \in \Psi} \bigvee_{r \in R_\psi} \bigwedge_{x \in \psi} L[x, r|_x].$$

This is a depth-3 formula, since the conjunction of conjunctions is still a conjunction.

**Proposition 4 (Completeness)** If $\Psi$ is satisfiable, then there is a satisfying assignment for $\Phi$, whose weight is $n \cdot D$.
Proof: Let $A : X \rightarrow \mathcal{F}$ be a satisfying assignment for $\Psi$. Define an assignment $A' : T \rightarrow \{\text{True, False}\}$ for the literals of $\Phi$ by setting $A'(T[x, \psi, a]) = \text{True}$ iff $A(x) = a$. This assignment clearly satisfies $\Phi$, and has weight exactly $nD$.

\[\text{Proposition 5 (Soundness)} \text{ If there is a weight-}g\text{ satisfying assignment for } \Phi, \text{ then there is an assignment satisfying } \frac{1}{2(2Dg)^D} \text{ fraction of } \Psi.\]

The proof of this proposition is very similar to the proof of Proposition 3.

Proof: Let $A_\Phi : T \rightarrow \{\text{True, False}\}$ be a weight-$g$ satisfying assignment for $\Phi$. For each variable $x \in X$, let $A(x) \overset{\text{def}}{=} \{a \in \mathcal{F} \mid A_\Phi(L[x, a]) = \text{True}\}$. $A(x)$ is non-empty since $x$ appears in some test $x \in \psi$, and for each $\psi \in \Psi$ there must be some $r$ for which $\wedge_{x \in \psi} L[x, r|x] = \text{True}$ because $A_\Phi$ satisfies $\Phi$.

$L[x, a]$ contains $|\Psi_x|$ literals that, if $a \in A(x)$, are by definition set to True. These are distinct for distinct $x$’s, thus

\[
\sum_{x \in A} |\Psi_x| \cdot |A(x)| \leq g \cdot nD.
\]

Consider the procedure of choosing a test $\psi \in R \Psi$ uniformly at random and then choosing a variable $x \in R \psi$ uniformly at random. The probability of choosing $x$ is $\frac{|\Psi_x|}{nD}$. The above equation is thus equivalent to $E(|A(x)|) \leq g$ where $E(|A(x)|)$ denotes the expectation of $|A(x)|$ where $x$ is chosen by the above procedure.

We call a variable $x$ for which $|A(x)| > 2D \cdot g$, a bad variable. The Markov inequality yields

\[
\Pr_x [|A(x)| > 2D \cdot E(|A(x)|)] < \frac{1}{2D}
\]

which means that the probability of hitting a bad variable is less than $\frac{1}{2D}$.

\[
\frac{1}{2D} \geq \Pr_{\psi \in \Psi, x \in \psi} [x \text{ is bad}] = \Pr_{\psi \in R \Psi} [\psi \text{ contains a bad variable}] \cdot \Pr_{x \in \psi} [x \text{ is bad | } \psi \text{ contains a bad variable}] \\
\geq \Pr_{\psi \in R \Psi} [\psi \text{ contains a bad variable}] \cdot \frac{1}{D}
\]

Multiplying by $D$, we deduce that at least half of the tests $\psi \in R \Psi$ contain no bad variable.

Next, we define a random assignment $A$ for $\Psi$ by choosing, for every variable $x$, a random value $a \in A(x)$, $A(x) \overset{\text{def}}{=} a$. For each test $\psi \in \Psi$ there is at least one value $r \in R_{\psi}$ with $\wedge_{x \in \psi} A_\Phi(L[x, r|x]) = \text{True}$ since $A_\Phi$ satisfies $\Phi$. The probability that each variable $x \in \psi$ was assigned $a = r|x \in A(x)$ is $\Pi_{x \in \psi} \frac{1}{|A(x)|}$. For tests that contain no bad variable, this probability is $\geq \frac{1}{(2D \cdot g)^D}$. Hence there is an assignment that satisfies at least

\[
\frac{1}{2} \cdot \frac{1}{(2D \cdot g)^D}
\]

fraction of the tests.

Since $\frac{1}{(2D \cdot g)^D} > \frac{1}{|\mathcal{F}|}$, we deduce that $\Psi$ is not a ‘no’ PCP instance.

We saw in Proposition 4 that if $\Psi$ is a PCP ‘yes’ instance then there is a weight-$nD$ satisfying assignment for $\Phi$. On the other hand, if $\Psi$ was a PCP ‘no’ instance (i.e. any assignment satisfies
no more than $1/|\mathcal{F}|$ fraction of the tests), then there cannot be even a weight-$gnD$ satisfying 
assignment for $\Phi$. Otherwise Proposition 5 would imply that there is an assignment satisfying 
$1/2 \cdot (2Dg)^D > 1/|\mathcal{F}|$ fraction of the tests (the last inequality follows mainly because $c' > c$). 
Thus, distinguishing between the case where the monotone formula has a satisfying assignment 
of weight $nD$ or $gnD$ is NP-hard because it enables distinguishing between ‘yes’ and ‘no’ PCP 
instances. This completes the proof of the theorem. ■

4 Reducing MMSA$_3$ to LABEL-COVER

In this section we show a reduction from MMSA$_3$ to LABEL-COVER. This shows that MMSA$_3$ 
is no-harder than LABEL-COVER, and (together with the reduction from [ABMP98]) places 
LABEL-COVER between level 3 and 4 in the ‘MMSA-hierarchy’. It also re-establishes the result 
in section 2 showing NP-hardness for approximating LABEL-COVER to within the same factor.

An instance of MMSA$_3$ is a formula

$$\Phi \overset{\text{def}}{=} \bigwedge_{i=1}^{I} \bigvee_{j=1}^{J} \bigwedge_{k=1}^{K} T_{i,j,k}$$

where the $T_{i,j,k}$ are literals from the set $\{x_1, ..., x_L\}$ for some $L \leq I \cdot J \cdot K$ (by repeating literals 
we may assume wlog that all conjunctions are of the same size, and similarly all disjunctions).

We construct a bipartite graph $G = (U, V, E)$ with vertices $U \overset{\text{def}}{=} \{u_1, ..., u_L\}$ for the literals, and 
$V \overset{\text{def}}{=} \bigcup_{w=1}^{W} \{v_{1,w}, ..., v_{L,w}\}$ for $W$ copies of the $I$ disjunctions (where $W$ is chosen large enough, 
say $W = L$). The edges in $E$ connect every literal to the disjunctions in which it appears.

$$E \overset{\text{def}}{=} \{(u_i, v_{i,w}) \mid \exists j, k, T_{i,j,k} = x_l\}$$

The sets of possible labels are $B_1 \overset{\text{def}}{=} \{0, 1, ..., W\}$ for $U$ and $B_2 \overset{\text{def}}{=} \{1, ..., W\}$ for $V$.

For $j = 1, ..., J$, denote $T_{i,j} = \{T_{i,j,k} \mid 1 \leq k \leq K\}$. If a vertex $v = v_{i,w}$ is labelled by $j$, we 
differentiate between two kinds of neighbors $u_l$ of $v$: those with $x_l \in T_{i,j}$ and those with $x_l \not\in T_{i,j}$. 
For an edge $e = (u_l, v_{i,w})$, we construct the relation $\Pi_e$ so that the two kinds of neighbors are 
‘covered’ differently.

$$\Pi_e \overset{\text{def}}{=} \{(w, j) \mid x_l \in T_{i,j}\} \cup \{(0, j) \mid x_l \not\in T_{i,j}\}.$$ 

Note that for every label $j$ for $v$ there is at least one $u_l$ for which $x_l \in T_{i,j}$, thus labelling $u_1, ..., u_L$ 
with 0 cannot be a total-cover.

**Proposition 6 (Completeness)** If there is a satisfying assignment for $\Phi$ with weight $t$, then 
there is a total-cover for $G$ with $l_1$-cost $L + t \cdot W = (t + 1) \cdot W$.

**Proof:** Let $A$ be a weight-$t$ satisfying assignment for $\Phi$. Define a cover as follows, for every 
$u_l \in U$ set

$$f_1(u_l) \overset{\text{def}}{=} \begin{cases} 
\{0, 1, ..., W\} & A(x_l) = \text{True} \\
\{0\} & \text{otherwise}
\end{cases}.$$
For every \( v_{i,w} \in V \) let \( f_2(v_{i,w}) \overset{def}{=} \{ j^* \} \) where \( j^* \) is the smallest index for which \( \bigwedge_{k=1}^{K} A(T_{i,j^*,k}) = \text{True} \) (such an index \( j^* \) exists because \( A \) satisfies \( \Phi \)). Obviously \( f_1, f_2 \) are non-empty, and the \( l_1 \) cost of the labelling is exactly \( L + t \cdot W \).

Let us show that the labelling \((f_1, f_2)\) is a total cover. Let \( e = (u_l, v_{i,w}) \) be an arbitrary edge, and let \( j \in f_2(v_{i,w}) \). By definition of \( f_2 \), \( j \) is such that \( A(x_l) = \text{True} \) for all \( x_l \in T_{i,j} \). Thus, for an index \( l \) with \( x_l \in T_{i,j} \), by definition \( f_1(u_l) = \{0,1,\ldots,W\} \) and \( e \) is covered by \((w,j)\). If \( x_l \not\in T_{i,j} \) then \((0,j) \in \Pi_e \) so \( e \) is covered because \( 0 \in f_1(u_l) \).

**Proposition 7 (Soundness)** If there is a total-cover for \( G \) with \( l_1 \)-cost \( g \cdot tW \), then there is a satisfying assignment for \( \Phi \) with weight \( gt \).

**Proof:** Let \((f_1, f_2)\) be a total cover with \( l_1 \) cost \( gt \cdot W \). Since \( \forall u \in U \quad f_1(u) \subseteq \{0,1,\ldots,W\} \), and \( \sum_{u \in U} |f_1(u)| = gt \cdot W \), there must be at least one \( w^* > 0 \) for which \(|\{u \mid w^* \in f_1(u)\}| \leq gt \). We claim that the assignment \( A \) (whose weight cannot exceed \( gt \)) defined by assigning \( x_l \) the value \text{True} if and only if \( w^* \in f_1(u_l) \), satisfies \( \Phi \).

Fix \( i \). We will show that the \( i \)th disjunction is satisfied. Consider the vertex \( v_{i,w^*} \) and a label \( j \in f_2(v_{i,w^*}) \neq \phi \). As before, define \( T_{i,j,k} = \{T_{i,k,j} \mid k = 1,\ldots,K\} \). We will show that the \( j \)th conjunction of the \( i \)th disjunction is satisfied (thus satisfying the whole disjunction). For this purpose we need to show that every literal \( x_l \in T_{i,j} \) is assigned \text{True}, or in other words \( w^* \in f_1(u_l) \). This is immediate since there is no other way of covering the edges \( e \overset{def}{=} (u_l, v_{i,w^*}) \).

Summarizing Propositions 6 and 7, we see that if the original formula \( \Phi \) had a satisfying assignment of weight \( t \), then the LABEL-COVER instance has a total-cover whose \( l_1 \)-cost is \( W(t+1) \). If, on the other hand, every satisfying assignment for \( \Phi \) has weight \( > gt \), then every total-cover has \( l_1 \)-cost \( > g \cdot tW \). This completes the reduction.

Choosing \( g = g_c(n) \) and by the result in the previous section we deduce that it is NP-hard to approximate LABEL-COVER to within a factor of \( \frac{gW}{W(t+1)} \geq g/2 = \Omega(2^{\log^{1-1/D}n}) \) where \( D = \log \log^c n \) for any \( c < 1/2 \). The proof for other \( l_p \) norms follows similarly.

## 5 Discussion and Open Questions

### A Depend-2 PCP Characterization of NP

In [ABSS93] LABEL-COVER was used to prove the hardness of the CLOSEST-VECTOR problem along with several other problems. However, they used a slightly modified version of LABEL-COVER, in which the relation \( \Pi_e \) for each edge is actually a function from \( B_1 \) to \( B_2 \). In our result, \( \Pi_e \) is a function from \( B_2 \) to \( B_1 \) and inherently cannot be extended to this version. This obstacle could be overcome had we known a low error-probability PCP characterization of NP with exactly two provers (i.e. a PCP test-system where each tests accesses exactly two variables, called depend-2-PCP). Compare this to the known low error-probability PCP characterization of NP [RS97, DFK99] where each test depends on a constant (\( > 2 \)) number of variables. Whether or not such a characterization exists remains an open question. Note that it is highly unlikely that this problem is in \( P \) since such an interactive proof protocol for \( NP \) exists [LS91, FL92, Raz98], with a quasi-polynomial blow-up.
The MMSA Hierarchy

We considered a hierarchy of approximation problems, equivalent to that in [GM97]. We showed a new hardness-of-approximation result for it (starting from the third level). Are higher levels in this hierarchy even harder to approximate, perhaps to within some polynomial $n^c$ factor? Such a result would immediately strengthen the known hardness results for the aforementioned problems in [GM97, ABMP98].

We know that Label-Cover resides between levels 3 and 4 in this hierarchy. However, the factor for which it is NP-hard to approximate Label-Cover is the same as for MMSA$_i$ for $i \geq 3$. Is this an indication that the hierarchy collapses, or is there really a difference in the hardness of hierarchy levels for $i \geq 3$?

References


