Lecture 2

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In this talk we will define "agreement expansion" and show it is present in the PCP constructions.

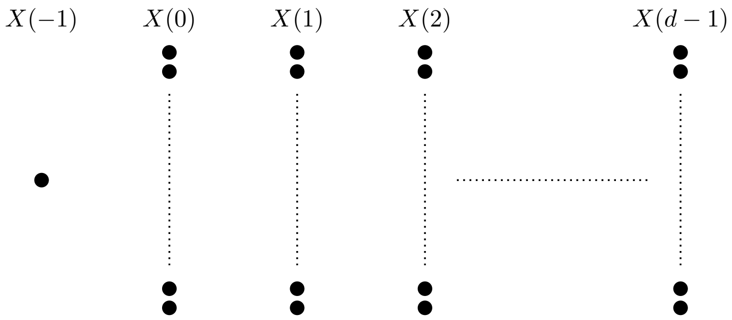

Figure 1: X is a dimension d expander, $X(i)$ contains the i-cells of X.

Let X be a $(k-1)$-simplicial complex, a function $f_0 \in \mathfrak{a}_0(X, \Sigma)$, i.e., $f_0 : X(0) \to \Sigma$ is a coloring of the vertices of X using colors from Σ. A function $f_i \in \mathfrak{a}_i(X, \Sigma)$ is a coloring of the subsets of vertices of size $(i+1)$, i.e $f_i(S)$ is a coloring of the vertices of S by colors from Σ.

First, let us look at the following example, let K_4 be the clique on four vertices, and let X be the full simplicial complex on four vertices. If we color the vertices of the graph by three colors, there will be a monochromatic edge, and we can expand such a coloring to the simplicial complex X by the coboundary operator. Alternatively, we can define a coloring on $X(1)$ such that for each $S \in X(1)$, the two vertices of the edge gets different colors, and such a coloring will not be consistent.

Definition 1. *Let $\mathfrak{a}(X, \Sigma)$ be the collection of all assignments, where f is an assignment if $\forall S \in X$, $f(S)$ is a S-coloring. We defined α as follows:*

$$\alpha_{t-1,k-1}(f) = \Pr_{S_1,S_2,T}[f(S_1)\mid_T = f(S_2)\mid_T].$$

Where T is chosen from $X(t-1)$ uniformly, and S_1 and S_2 are chosen uniformly and independently from $X(k-1)$ such that $T \subset S_1, S_2$.

If $f_0 \in \mathfrak{a}_0(X, \Sigma)$, and $f = U^i f_0$ then for every $i \in [k-1]$, we have $\alpha(f) = 1$, where $\alpha(f) = \min_t \alpha_{t-1,k-1}(f)$.

Definition 2. *Let X be a $k-1$ dimensional simplicial complex, and let $S \in X(i)$. The link of S, $X_S = \{S' \setminus S \mid S' \in X, S \subset S'\}$.*

X_S is a $(k-1-|S|)$-dim simplicial complex, and we denote by $X_S(j)$ it's j^{th}.

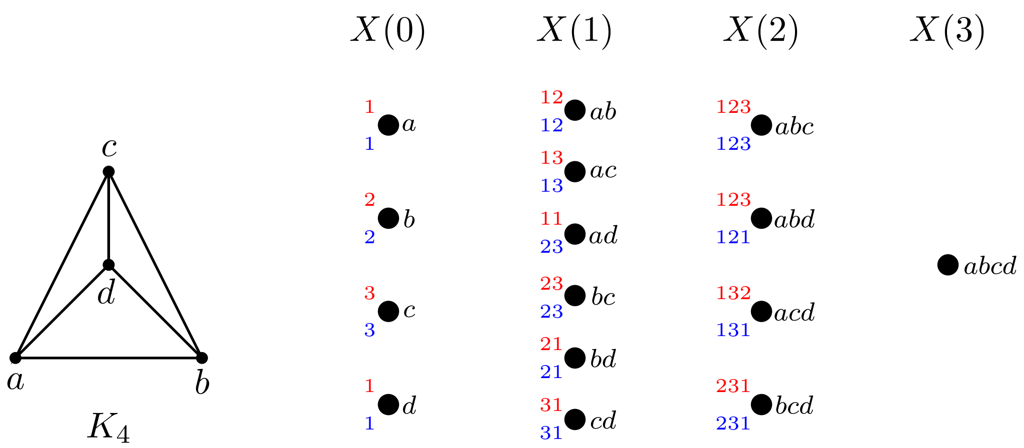

Figure 2: X is K_4, with all of the triangles and a single 4-cell, K_4 is not 3 colorable. The red coloring is a coloring of $X(0)$, extended in a consistent way to $X(1), X(2)$. The blue coloring colors all of the vertices in each i-cell differently, but is not consistent.

Claim 3. *If $X = \Delta_{k-1}$, $f \in \mathfrak{a}_{k-1}$, and $\alpha_{k-2,k-1}(f) = 1$ then there exists f_0 such that $f = U^{k-1} f_0$.*

Proof. For each $u \in X(0)$, define $f_0(u) := f(S \cup \{u\}) \mid_u$ for some $S \in X_u(k-1)$. If there are $S_1, S_2 \in X_u(k-2)$ such that $f(S_1 \cup \{u\}) \neq f(S_2 \cup \{u\})$ then there is a "walk" $S_1, T_1, \ldots, T_\ell, S_2$, such that for all the consecutive pairs they intersect on one dimension less, and thus must agree (since $\alpha(f) = 1$). All of $T_1, \ldots T_\ell$ are in X since it is the full simplcial complex. □

Definition 4. *The agreement expansion of X, denoted by $\gamma_{t-1,k-1}$ is defined as follows:*

$$\gamma_{t-1,k-1} = \min_{f \in \mathfrak{a}_{k-1} \setminus u^{k-1}\mathfrak{a}_0} \frac{1 - \alpha_{t-1,k-1}(f)}{dist(f, U^{k-1}\mathfrak{a}_0)},$$

where $U^{k-1}\mathfrak{a}_0 = \{U^{k-1} f_0 \mid f_0 \in \mathfrak{a}_0\}$ and $dist_{X(k-1)}(f, g) = \Pr_{S \in X(k-1)}[f(S) \neq g(S)]$.

Definition 5. *Fix $0 < p < 1$, and let $f \in \mathfrak{a}_{k-1}(X, \Sigma)$. We define $\alpha_p(f) = \Pr[f(S_1)_T = f(S_2)_T]$, where S_1, S_2, and T are chosen by the following process:*

1. *Select $S_1 \in X(k-1)$ uniformly.*
2. *Remove each $u \in S_1$ with probability p and obtain T.*
3. *Select $S' \in X_{k-1-|T|}$, and output S_1, T, and $S_2 = S' \cup T$.*

Claim 6.

$$\alpha_p(f) = \sum_{i=0}^{k} \binom{k}{i} p^i (1-p)^{k-i} \alpha_{i-1,k-1}(f),$$

where $\alpha_{-1,k-1} = 1$.

Our goals here would be to study the above α's. More precisely, given X, and $f \in \mathfrak{a}_{k-1}$ if $\alpha(f) > 1 - \delta$, what does it imply for f? We could also ask if X is a complex for which $\alpha(f) > 1 - \delta$ implies "structure" of f (for example $f \approx U^{k-1}\mathfrak{a}_0$), what does it mean about X? Next we will see the relation between agreement expansions and PCP constructions.

Theorem 7 (PCP Theorem)**.** *There is a polynomial time algorithm taking an input graph G and outputting a graph H such that if val$(G) = 1$ then val$(H) = 1$ and if val$(G) < 1$ then val$(H) < 0.999$, where val$(G) = \max_{C:V\to\{1,2,3\}}\{\Pr_{(u,v)\in E}[C(u) \neq C(v)]\}$, the maximum fraction of non-monochromatic egdes over all possible colorings over the vertices of G.*

A key intermediate step in proving the above result is to move from a graph G to a complex X (and later we would move from X to H). Elaborating, G would be a 3-coloring problem and X would be an agreement problem.

We want to build a simplcial complex over G, that will contain the edges of G. The graph G can be sparse, so taking all the (or random) subsets will cause almost all of the subsets in X to be empty. Instead, for every vertex $v \in V(G)$, we construct a "ball" $B_{v,r} = \{u \in V(G) \mid \text{dist}_G(u,v) \leq r\}$. We define X as follows: $X(0) = V$, and $X(r_i)$ contains the balls of radius r_i for every $v \in V$, for $i = \{1,2\}$. The resulting X is not a simplcial complex, since not all subsets of $S \in X(r_1)$ are in X.

We will assume that G is an expander, if the original graph wasn't an expander we can transform it into one easily. Given $f \in \mathfrak{a}(X)$, we would like to determine $\alpha(f)$. We define the following:

$$\mathfrak{a}_i^\star(X,\Sigma) = \{f \mid \forall S \in X(r_i), f(S) \text{ respects the local constraints in } S\}.$$

We note that this set might be empty, if K_4 is in G, but we are not interested in such graphs, since deciding 3-coloring on them is easy.

We define $\text{val}(X) := \max_{f\in\mathfrak{a}_2^\star(X,\Sigma)} \alpha(f)$. Where in the distribution in α we pick a uniform vertex $u \in X(r_2)$, preform a random walk of length about r_2, choose v to be the end vertex and w to be a vertex in the middle. Then $S_1 = B_{u,r_2}$, $S_2 = B_{v,r_2}$ and $T = B_{w,r_1}$.

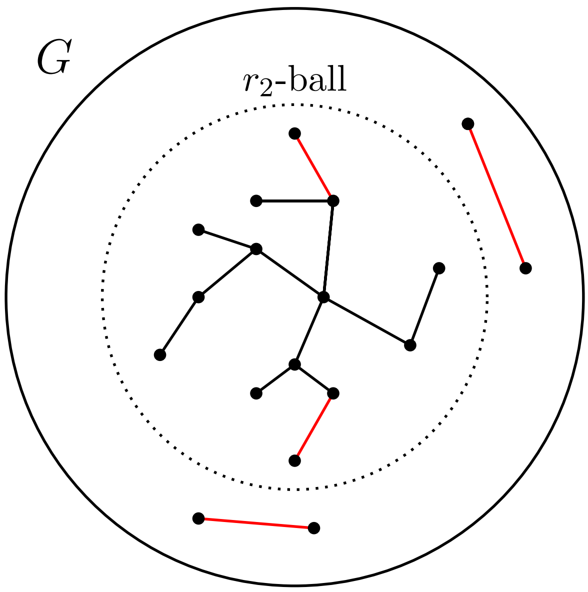

Figure 3: A ball of radius r_2 around a vertex $u \in V$, in red are the edges that the "best" coloring c violates.

We now would like to show that if $\text{val}(G) = 1$ then $\text{val}(X) = 1$ and if $\text{val}(G) < 1 - \delta$ then $\text{val}(X) < 1 - 2\delta$.

If $\text{val}(G) = 1$ then there exists $c : V \to \{1, 2, 3\}$ a perfect coloring, $U^2 c \in \mathfrak{a}_2^\star(X, \{1, 2, 3\})$ then, $\text{val}(X) = 1$ because $\alpha(U^2 c) = 1$. On the other hand if $\text{val}(G) = 1-\delta$ ($\delta > 0$) then we fix a "best" $c : V \to \{1, 2, 3\}$, $f = U^2 c$. Clearly, $f \notin \mathfrak{a}_2^\star$. Moreover, f is $r_2\delta$ far from $\mathfrak{a}_2^\star$ (because the underlying graph G was an expander). If $f \in \mathfrak{a}_2^\star$, then we would like to show that $\alpha(f) > 1-2\delta$, and the idea is to show that there exists some f_0 such that $U^2 f_0 \approx f$, and $\text{val}(f_0) > 1 - \delta$.