Lecture 4

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Today, we will see the proof of *a* theorem.

Theorem 1. *Fix* $\Sigma = \{0,1\}$. *The complete complex (k-dimensional, n-vertices) is an agreement expander, i.e.,* $\exists c > 0,\ \gamma_{\frac{1}{2}}(X(n,k)) > c$.

We recall from the diagram above that $\mathfrak{a}_0 = \{g : X(0) \to \Sigma\}$, $U^{k-1}g(S) = g|_S \in \Sigma^k$ and $\mathfrak{a}_{k-1} = \{f : X(k-1) \to \Sigma^k\}$. Given $f \in \mathfrak{a}_{k-1}$, define its agreement parameter as follows:

$$\alpha_{t-1,k-1}(f) \triangleq \Pr_{T \in_R X(t-1), T \subset S, S' \in X(k-1)} [f(S)_T = f(S')_T].$$

Additionally, we have a different agreement parameter defined as follows:

$$\alpha_p(f) \triangleq \Pr_{T,S,S'}[f(S)_T = f(S')_T] = \sum_{i=0}^{k} \binom{k}{i} p^i (1-p)^{k-i} \alpha_{k-1,i-1}(f).$$

Where $S \in_R X(k-1)$, T is chosen by removing each element from S with probability p, and $S' \in X(k-1)$ is chosen by choosing a uniform element in $X(k-1)$ containing T.

Next, we restate Theorem 1, as follows:

Theorem 2. *If* $\alpha_{\frac{1}{2}}(f) > 1 - \delta$ *then* f *is close to* $U^{k-1}\mathfrak{a}_0$, *i.e., there exists* $g : X(0) \to \{0,1\}$ *such that the following holds:*

$$\Pr_{S \in X(k-1)}[f(S) = U^{k-1}g(S)] \geq 1 - c\delta$$

It is an open question to find a sparse (i.e., bounded degree) complex for which the same holds. A weak version of Theorem 2 that is easier to prove is the following:

Lemma 3. *If* $\alpha_{\frac{1}{2}}(f) > 1 - \delta$ *then dist*$(f, U^{k-1}\mathfrak{a}_0) < k\delta$.

A similar lemma is true for Ramanujan complexes, and even for random sparse complexes. But this is not interesting for hardness amplification. Nonetheless, we see the proof of the lemma. In a sparse random simplicial complex $\alpha_{\frac{1}{2}}$ is meaningless, since with high probability there is only a single S that contains T. Instead, we may use α_0, in which T is always a single element, $T \in X(0)$.

Proof of Lemma 3. Given f as in the lemma we define $g : X(0) \to \{0,1\}$ to be the plurality value of f:

$$\forall x, g(x) = \operatorname*{argmax}_{\beta \in \{0,1\}} \Pr[f(S)_x = \beta].$$

We have the following claim:

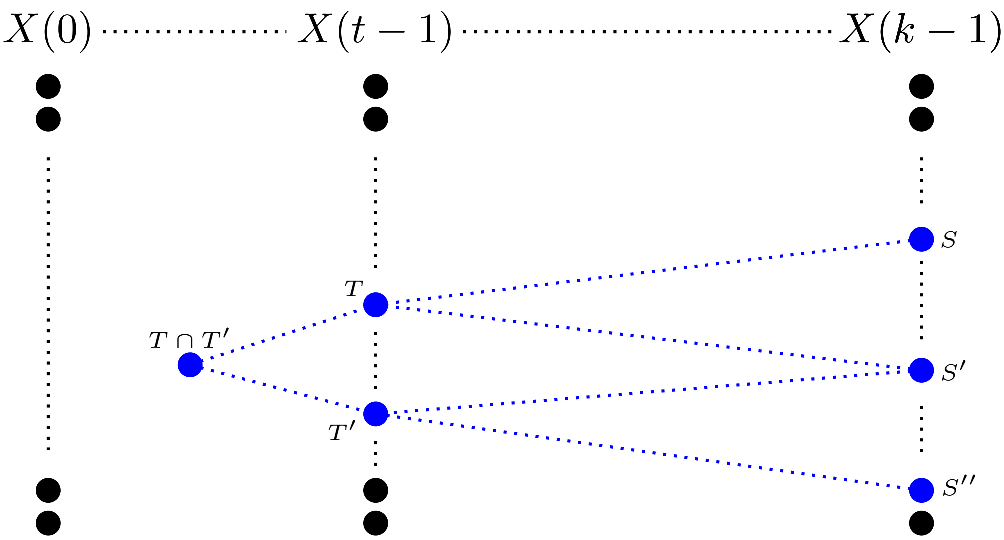

Figure 1: On the random process between S and T we drop each element in S with probability p, and then go back to the $k-1$ layer, so $\mathbb{E}[|S \cup S'|] = pk$. If we do the same thing from S' getting S'', then $\mathbb{E}[|S \cup S''|] = p^2 k$. Note that T, T' may not be in the same layer.

Claim 4.

$$\alpha_{p^2}(f) \geq \alpha_p(f)^2.$$

Proof. This follows by noticing that it is precisely capturing two steps in the p-random walk. □

From this, we can deduce the following:

Corollary 5. *If $\alpha_{\frac{1}{2}}(f) \geq 1 - \delta$ then $\alpha_0(f) \geq 1 - \delta \log k$.*

We remark here that $\alpha_0(f)$ is computed from intersections of size equal to 1. The lemma follows by rephrasing it as follows: if $\alpha_0(f) \geq 1 - \delta'$ then $\mathrm{dist}(f, U^{k-1}\mathfrak{a}_0) \leq k \cdot \delta'$. This is because, by definition of g we have that $\Pr_{x \in S}[g(x) = f(S)_x] \geq (1-\delta')^2 \geq 1 - 2\delta'$. By union bound, S agree with $1 - \delta'$ of its x's, thus with probability $1 - |S|\delta'$, it agrees with all of its elements (S "agrees" with x if and only if $f(S)_x = g(x)$). □

We will now talk about the tightness of Theorem 2. In particular, there exists a function $f \in \mathfrak{a}_{k-1}$, such that $\alpha_{k-2}(f) > 1 - \delta/(k/2)$ but f is far from $U^{k-1}\mathfrak{a}_0$, i.e., $\forall g \in \mathfrak{a}_0, \Pr_S[f(S) = U^{k-1}g(S)] \leq \exp(-k)$. We construct such a f as follows: fix an order on $X(0) = [n]$. Further, fix functions, $g_1, \ldots g_{n-k} : [n] \to \{0,1\}$. We write $X(k-1) = Y_1 \cup Y_2 \cup \cdots Y_{n-k}$ and let $f(S) = U^{k-1}g_i(S)$ if $S \in Y_i$. We set $Y_i = \{S \in X(k-1) \mid \min(x \in S) = i\}$. Then $Y_1 = \{1 \in S\}$, $Y_2 = \{2 \in S\} \setminus Y_1$, and so on. A simple calculation shows that $\alpha_{k-2}(f) \geq 1 - \frac{c}{k}$.

Let $g \in \mathfrak{a}_0$, since all the functions g_i are different, we assume that their value on an input is different with probability 2. Then $\Pr_S[f(S) = U^{k-1}g(S)] \leq \Theta(\frac{1}{n}) + 2^{-k}$, since the probability of g to hit the correct function g_i on which $f(S) = g_i(S)$ is about $\frac{1}{n}$, and else the probability of equality ia 2^{-k}. In our setting $n \gg k$, so the construction works.

Proof of Theorem 2. We use Claim 4, and the probabilistic argument to obtain that $\alpha_{\frac{k}{10}} > 1 - \delta$.

Let $A \in X(\frac{k}{20})$, $\sigma \in \{0,1\}^A$. We define $L(A)$ to be the set of all sized k cofaces of A and $L(A, \sigma)$ be the set of all such cofaces whose assignment to A is σ. We say that $S \sim S'$ if $|S \cap S'| = k/10$. As in previous lectures, we define good and excellent simplices as follows.

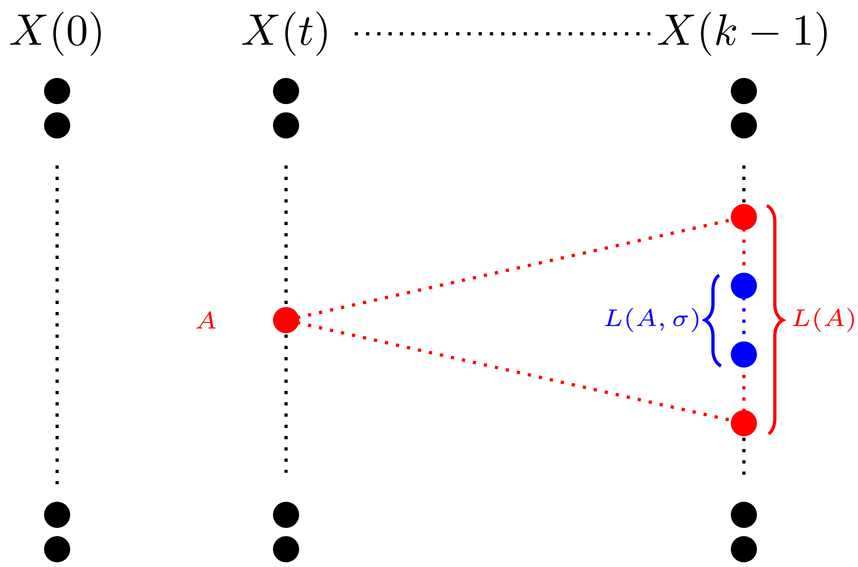

Figure 2: For each $A \in X(t)$, we define $L(A)$ to be the k sets containing A and $L(A, \sigma)$ to be those that also satisfy $f(S)_T = \sigma$.

Definition 6. *A is good if the following holds:*

$$\Pr_{S \sim S'}[f(S)_{S\cap S'} = f(S')_{S\cap S'} \mid A \subseteq S \cap S'] \geq \frac{\alpha_{\frac{k}{10}}(f)}{2} > \frac{1-\delta}{2}.$$

Definition 7. *(A, σ) is excellent if the following holds:*

$$\Pr_{S \sim S'}[f(S)_{S\cap S'} = f(S')_{S\cap S'} \mid S, S' \in L(A, \sigma)] \geq 1 - \textit{tiny}.$$

The proof sketch then follows by first showing that many As are good, and that a good A is also excellent with high probability. Next, we would define the majority function $g_{A,\sigma} : X(0) \to \{0, 1\}$ on excellent (A, σ) and do the following:

1. Prove that $\Pr_S[f(S) = U^{k-1} g_{A,\sigma}(S) \mid S \in L(A, \sigma)] \geq 1 - O(\delta)$.
2. Stitch together $g_{A,\sigma}$ over many As and σs.

□