Lecture 1

Lecturer: Tali Kaufman
Preliminary Scriber: Karthik C. S.
Scribe Upgraders: Inbal Livni and David Mass

1 Introduction

Expander graphs are very useful for many applications in Computer Science, such as the construction of good error correcting codes [SS96] or Dinur's proof of the PCP theorem [Din07]. Though, there are a few applications for which they are not good enough– We cannot get locally testable codes from them, and the PCPs which are constructed from expanders are not strong enough. The motivation then is that high dimensional expanders might take us further and give us these applications as well.

We start with some general notes about high dimensional expanders. First, as opposed to graphs, where a random d-regular graph is an expander with high probability, in higher dimension it does not hold. This implies that they are harder to get. Second, as expander graphs are sparse graphs that resemble the behavior of the complete graph, with high dimensional expanders we want sparse objects that resemble the complete d-dimensional hypergraph. And last, while expander graphs capture only pairwise interactions, high dimensional expanders capture k-wise interactions.

In this talk we will introduce the notion of high dimensional expansion, including some background in cohomology. In the next talk we will understand high dimensional expansion in the context of property testing. In the third talk we will see an explicit construction of objects which are candidates for high dimensional expanders, namely, the Ramanujan complexes which generalize the Ramanujan graphs of [LPS88]. In the final two talks we will show that Ramanujan complexes are indeed high dimensional expanders.

2 Expander graphs

We recall some basic properties of expander graphs. Let $G = (V, E)$ be a k-regular, connected, non-bipartite graph. There are two notions of expansion:

1. Strong connectivity, or *combinatorial expansion*. It is measured by the Cheeger constant of the graph, which is defined as

$$h(G) = \min_{\substack{S \subseteq V \\ S \neq \emptyset, V}} \frac{|E(S, \bar{S})|}{\min(|S|, |\bar{S}|)}.$$

   If $h(G) \geq \varepsilon$, then G is called an ε-combinatorial expander.

2. *Spectral expansion*. Denote by A_G be the adjacency matrix of G, and let

$$\lambda(G) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } A_G,\ \lambda \neq \pm d\}.$$

If $\lambda(G) \le d - \varepsilon$, then G is called an ε-spectral expander.

Lemma 1 (Expander mixing lemma)**.**

$$\forall S, T \subseteq V, \qquad \left| |E(S,T)| - \frac{d|S||T|}{|V|} \right| \le \lambda(G)\sqrt{|S||T|}.$$

For graphs, combinatorial expansion and spectral expansion are approximately equivalent. Their relation is given by the Cheeger inequality.

Lemma 2 (Cheeger inequality)**.**

$$h(G) \ge \frac{d - \lambda(G)}{2},$$

and if $\tilde{\lambda}(G) = \max\{|\lambda| \mid \lambda$ *is an eigenvalue of* A_G, $\lambda \neq d\}$, *then*

$$\frac{d - \tilde{\lambda}(G)}{2} \le h(G) \le \sqrt{2d(d - \tilde{\lambda}(G))}.$$

3 Cohomology over $\mathbb{F}_2$

A simplicial complex X is a set system with a closure property, namely, for any $F \in X$ all of its subsets $G \subseteq F$ satisfy $G \in X$. Each set $F \in X$ is called a face of the complex. The dimension of a face $F \in X$ is defined as $\dim(F) = |F| - 1$. The faces of dimension i are denoted by $X(i) = \{F \in X \mid \dim(F) = i\}$, so $X(0)$ are the vertices, $X(1)$ are the edges, $X(2)$ are the triangles, etc. $X(-1) = \{\emptyset\}$ is also a face of the complex. The dimension of the complex is $\dim(X) = \max_d\{X(d) \neq \emptyset\}$. The complex is called pure if all the maximal faces are of the same dimension d, i.e., for any face $F \in X$ with $\dim(F) < d$, there exists a face $G \in X(d)$ such that $F \subset G$. For the rest of the talk we will assume that the complex is pure.

Definition 3 (Cochains)**.** *An i-cochain over $\mathbb{F}_2$ is a function $f : X(i) \to \{0,1\}$. The space of all i-cochains is*

$$C^i(X) = C^i(X, \mathbb{F}_2) = \{f : X(i) \to \{0,1\}\}.$$

For example, $C^0(X)$ are all subsets of vertices, $C^1(X)$ are all subsets of edges, $C^2(X)$ are all subsets of triangles, etc. $C^{-1}(X)$ are the functions on a single element, which is the empty set, so $C^{-1}(X) = \{0,1\}$.

Definition 4 (Coboundary map)**.** *The i-coboundary map $\delta_i : C^i \to C^{i+1}$ maps a function on the i-faces to a function on the $(i+1)$-faces, and is defined by*

$$(\delta_i f)(G) = \sum_{\substack{F \subseteq G \\ |F| = |G|-1}} f(F),$$

where $f \in C^i(X)$ and $G \in X(i+1)$.

For example, if f is a function on edges, then $\delta_i f$ is a function on triangles such that each triangle gets the sum of the edges contained in it (mod 2).

In a similar way we have the boundary map which maps functions to the other direction.

Definition 5 (Boundary map)**.** *The i-boundary map $\partial_i : C^{i+1} \to C^i$ maps a function on the $(i+1)$-faces to a function on the i-faces, and is defined by*

$$(\partial_i f)(G) = \sum_{\substack{F \supseteq G \\ |F|=|G|+1}} f(F),$$

where $f \in C^{i+1}(X)$ and $G \in X(i)$.

We focus on the coboundary map (going from i-cochains to $(i+1)$-cochains) and define the coboundaries and the cocycles of the complex.

Definition 6 (Coboundaries)**.** *The i-coboundaries are all i-cochains of the form $\delta_{i-1} f$ for some $f \in C^{i-1}(X)$, i.e.,*

$$B^i(X) = B^i(X, \mathbb{F}_2) = Im\,(\delta_{i-1}).$$

For example, $B^0(X) = \mathrm{Im}\,(\delta_{-1}) = \{\mathbf{0}, \mathbf{1}\}$, where $\mathbf{0}$ and $\mathbf{1}$ denote the constant all zeros and all ones functions. When viewing the 0-cochains as subsets of vertices we get that $B^0(X) = \{\emptyset, X(0)\}$. The 1-coboundaries are functions on edges which are derived from functions on vertices. Then we get that $B^1(X) = \mathrm{Im}\,(\delta_0) = \mathrm{CUTS}$, where a cut is the set of edges between a partition of the vertices of the complex into two disjoint subsets. This holds since any 0-cochain $f \in C^0(X)$ partition the vertices into two disjoint subsets, and each edge gets the sum of its two endpoints (mod 2).

Definition 7 (Cocycles)**.** *The i-cocycles are all i-cochains $f \in C^i(X)$ for which $\delta_i f = \mathbf{0}$, i.e.,*

$$Z^i(X) = Ker\,(\delta_i).$$

We note that $\delta_i \circ \delta_{i-1} f = \mathbf{0}$ for any i-cochain $f \in C^{i-1}(X)$. (Left as an exercise.) It follows that $B^i \subseteq Z^i$. It can be visualized by the following figure.

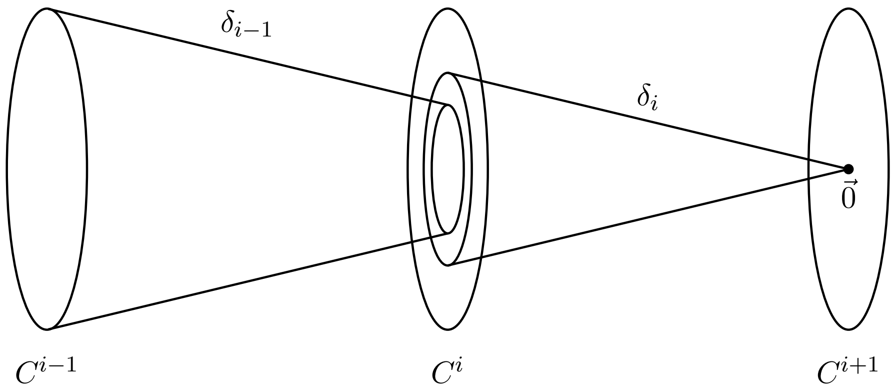

Figure 1: δ_i is a function from C^i to C^{i+1}, the image of δ_{i-1} (the i-coboundaries) is contained in the kernal of δ_i (the i-cocycles).

Definition 8 (Cohomology)**.** *The i-cohomology is defined as the quotient space $H^i(X) = Z^i(X)/B^i(X)$, i.e., it is a space of equivalent classes of i-cocycles where two i-cocycles are equal if they differ by a coboundary.*

Let us show how connectivity of graphs is related to the cohomolgy.

Example 9. Let $X = (V, E)$ be a simplicial complex of dimension 1 (i.e., it contains only vertices and edges so it is a graph). Any 0-cochain $f \in C^0(X)$ is of the form $f = \mathbf{1}_A$ (the characteristic function of A) for some subset of vertices $A \subseteq V$. The coboundary map is then $\delta_0 \mathbf{1}_A = \mathbf{1}_{E(A,\bar{A})}$, i.e., the set of edges between A and $\bar{A}$. It follows that $\mathbf{1}_A \in Z^0(X)$ if and only if A is a connected component in X, so $\dim(Z^0(X))$ equals the number of connected components in the graph. Also note that since $B^0(X) = \{\mathbf{0}, \mathbf{1}\}$, then $\dim(B^0(X)) = 1$. Then we get that $\dim(H^0(X)) = \dim(Z^0(X)) - \dim(B^0(X))$ is the number of connected components minus 1. In other words, $H^0(X) = 0$ if and only if the graph is connected.

As we have seen, $H^0(X) = 0$ means that the graph is connected. According to that we say that the cohomology is a measure for connectivity at any dimension, i.e., $H^i(X) = 0$ means that dimension i of X is connected for any i.

4 High dimensional expansion

We have defined connectivity for any dimension of the complex. Now we want to define a measure of strong connectivity.

Definition 10 (Strong connectivity)**.** *Let X be a d-dimensional simplicial complex. For any $i \in \{0, \ldots, d-1\}$ define*

$$\varepsilon_i(X) = \min_{f \in C^i(X) \setminus B^i(X)} \frac{\|\delta_i f\|}{\|dist(f, B^i(X))\|},$$

where $\|\delta_i f\| = \dfrac{|\delta_i f|}{|X(i+1)|}$ *and* $\|dist(f, B^i(X))\| = \min\limits_{b \in B^i(X)} \dfrac{|f+b|}{|X(i)|}$ *is the hamming distance of f from the coboundaries.*

Note that $\varepsilon_i(X) > 0$ if and only if $H^i(X) = 0$ (i.e., dimension i is connected). Moreover, larger $\varepsilon_i(X)$ means that the connectivity of dimension i is *stronger*. We can see this clearly for the case of graphs since

$$\varepsilon_0(X) = \min_{f \in C^0(X) \setminus B^0(X)} \frac{\|\delta f\|}{\|\mathrm{dist}(f, B^0(X))\|} = \min_{\substack{A \subseteq X(0) \\ A \neq \emptyset, X(0)}} \frac{\|E(A, \bar{A})\|}{\min(\|A\|, \|\bar{A}\|)} = h(G) \frac{|V|}{|E|},$$

where $G = (X(0), X(1))$ is the 1-dimensional skeleton of X.

Then we define high dimensional expansion as follows.

Definition 11 (Coboundary expansion)**.** *Let X be a simplicial complex of dimension d. X is called an ε-coboundary expander, $\varepsilon > 0$, if for any $i \in \{0, \ldots, d-1\}$, $\varepsilon_i(X) \geq \varepsilon$.*

The above definition of high dimensional expansion is then a strong notion of connectivity for high dimensional simplicial complexes. We could define it in different ways. We have the freedom to choose which coboundary map we work with, and which object we want to be far from. We have chosen to work over $\mathbb{F}_2$ since we view it as a set of tests, as will be described in the next talk. We note that different applications might require other definitions.

In summary, once defining high dimensional expansion we should ask ourselves:

1. What is the *coboundary operator*.

2. The coboundary should be large *with respect to what.*

We finish this lecture with a small discussion about the complete d-dimensional complex $\Delta_d = \binom{[n]}{d+1}$, i.e., the complex containing all faces of dimension d. The following theorem has been proven by Gromov, and is related to a notion called topological overlapping.

Theorem 12. *The complete complex of dimension d is an ε-coboundary expander with $\varepsilon > 1$.*

Currently, there is no known *bounded degree* complex which is an ε-coboundary expander with $\varepsilon > 0$, where a d-dimensional complex is called bounded degree if the number of d-faces containing any vertex is bounded by a constant, independent of the number of vertices in the complex.

References

[Din07] Irit Dinur. The pcp theorem by gap amplification. *Journal of the ACM (JACM)*, 54(3):12, 2007.

[LPS88] Alexander Lubotzky, Ralph Phillips, and Peter Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[SS96] Michael Sipser and Daniel A Spielman. Expander codes. *IEEE Transactions on Information Theory*, 42(6):1710–1722, 1996.