1 Co-systolic expansion

In the previous talk, we saw the construction of Ramanujan Complexes. We recapitulate a few details here. Given a Ramanujan complex $X$ of dimension $d$, for every $i \in \{0, \ldots, d-1\}$, the link of every $\sigma \in X(i)$ is the spherical building $\mathbb{P}^{d-1-i}(\mathbb{F}_p)$. (Recall that for $\sigma \in X(i)$, the link $X_\sigma$ of $\sigma$ is the set of all faces in $X$ that contain $\sigma$, after removing $\sigma$ from each of them; that is, $X_\sigma = \{ \tau \setminus \sigma \mid \sigma \subseteq \tau \in X \}$.)

Additionally, the degree of every vertex in the complex is bounded. This holds because for every $v \in X(0)$, we have that $X_v = \mathbb{P}^{d-1}(\mathbb{F}_p)$, and in particular the vertices of $X_v$ correspond to all the proper subspaces of $\mathbb{F}_p^d$. Therefore, the number of vertices in $X_v$ satisfies:

$$|X_v(0)| = \sum_{i=1}^{d-1} \binom{d}{i} p^{(d-i)},$$

which (since $d$ and $p$ are constants) is also a constant. Hence, the link $X_v$ also contains a constant number of faces, which implies that $\deg(v)$ is a constant. In contrast, the spherical building itself is not “bounded degree”, in the sense that the degree of the vertices in the spherical building is indeed a function of the number of vertices in it (i.e., the degree of every $\sigma \in X_v(0)$ is a function of $|X_v(0)|$; we leave this to the reader to verify).

Co-systolic expansion. In today’s talk, we will see that these complexes are co-systolic expanders. We recall the definition below.

**Definition 1.** Let $X$ be a simplicial complex of dimension $d$. $X$ is an $(\varepsilon, \mu)$-co-systolic expander if for every $0 \leq k < d$ the following holds:

$$\min_{A \in C^k \setminus B^k} \frac{\|\delta A\|}{\| \text{dist}(A, Z^k) \|} \geq \varepsilon \quad \text{and} \quad \forall A \in Z^k \setminus B^k, \|A\| \geq \mu.$$
For the moment, let us consider the norm of a cochain $A \in C^k$ to be $\|A\| = \frac{|A|}{|X(\{k\})|}$. We will actually need a slightly different definition for the norm, but for now this definition will provide sufficiently good intuition, and we will present the refined definition later on. Recall, from previous lectures, that the definition above is equivalent to the following definition: A complex $X$ is an $(\varepsilon, \mu)$-co-systolic expander if $Z^k$ is $(k + 2, \varepsilon)$-testable by the canonical test and every $A \in Z^k$ that is not “simple” (i.e., every $A \in Z^k \setminus B^k$) is $\mu$-large.

In the case of graphs, a graph that is a co-systolic expander is composed of large connected components where each component is an expander.

Why should we study co-systolic expansion? One motivation is that co-systolic expansion is somewhat analogous to the notion of small-set expansion in graphs; it can be seen as a small-set expansion that corresponds to the notion of co-boundary expansion (i.e., to the operator $\delta_i$ as we defined it).

In the case of Ramanujan complexes we know that they are not all co-boundary expanders, i.e., there is a subsequence of Ramanujan complexes that are not co-boundary expanders. However, we do know that a co-systolic expander that is also “connected” (in a certain formal sense) is a co-boundary expander. Hence, if we would find a subsequence of Ramanujan graphs that are connected, they would be co-boundary expanders.

Another motivation to study co-systolic expansion is its relation to the topological overlapping property, as we describe next.

## 2 Topological overlapping property

We will start by explaining this notion for graphs and then move to higher dimensions.

A graph is said to have the topological overlapping property, if for every embedding of its vertices on the real line, there exists $z \in \mathbb{R}$ that is covered by a constant fraction of the edges. It is known that every expander graph has the topological overlapping property (but not vice versa).

A 2-dimensional simplicial complex $X$ has the topological overlapping property, if for every embedding $f$ of $X(0)$ in $\mathbb{R}^2$ and for every continuous function $\tilde{f} : X \rightarrow \mathbb{R}^2$ (that extends $f$), there exists $z \in \mathbb{R}^2$ which is covered by a constant fraction of the triangles.

This definition can be extended to higher dimensions, where we note here that the requirement is always only for the maximal dimension in $X$ (for example, in dimension 2 the requirement is only on the triangles and not on the edges). The following results, which have been proven by Gromov [Gro10], link expansion in high dimensions with the topological overlapping property.

**Theorem 2.** Co-boundary expansion implies the topological overlapping property.

**Theorem 3.** Co-systolic expansion implies the topological overlapping property.

## 3 Proof that Ramanujan complexes are co-systolic expanders

The main result that we wish to prove is, informally, the following:

**Theorem 4 (Informal).** A complex with “nice links” is a co-systolic expander.

Recall that in Ramanujan complexes the links are spherical buildings, which are indeed “nice”. Hence, it will follow that Ramanujan complexes are co-systolic expanders.
3.1 The refined definition of the norm

Let us first present a refined definition for the norm \( \|A\| \) of a cochain \( A \). The reason that we need a definition different from the one above is that we need to work with non-regular complexes (because Ramanujan complexes are not regular in every dimension).

**Definition 5** (Norm). Let \( X \) be a \( d \)-dimensional complex, and let \( 0 \leq k \leq d \). For every \( \alpha \in X(k) \), the degree of \( \alpha \), \( \deg(\alpha) \), is the number of \( d \)-cells containing \( \alpha \). We define the normalized degree \( w(\alpha) \) as follows:

\[
w(\alpha) = \frac{\deg(\alpha)}{\sum_{\tau \in X(k)} \deg(\tau)}.
\]

Now, for any cochain \( A \in C^k(X) \), the norm of \( A \) is defined to be \( \|A\| = \sum_{\alpha \in A} w(\alpha) \).

An equivalent definition of this norm suggested by Roei Tell as follows. We look at the following random process: Choose uniformly at random a face \( P \in X(d) \), and iteratively, for \( d \) steps, remove an element from \( P \) uniformly at random. We obtain the random variable \( P = (P_d, P_{d-1}, \ldots, P_0) \). Then, for every \( 0 \leq k \leq d \), and \( \alpha \in X(k) \), it holds that \( w(\alpha) = \Pr[P_k = \alpha] \); and for every \( A \in C^k(X) \) it holds that \( \|A\| = \Pr[P_k \in A] \).

An additional important concept is the norm of a cochain inside a link. For \( i < k \leq d \), given \( A \in C^k(X) \) and \( \sigma \in X(i) \), we define the localization of \( A \) to the link of \( \sigma \) as \( A_{\sigma} = \{ \tau \setminus \sigma \mid \sigma \subseteq \tau \in A \} \). Then, \( \|A_{\sigma}\|_{\sigma} \) is simply the norm of \( A_{\sigma} \) inside the link \( X_{\sigma} \) (i.e., \( \| \cdot \|_{\sigma} \) is the norm defined in the complex \( X_{\sigma} \)). Using the equivalent definition above, we have that \( \|A_{\sigma}\|_{\sigma} = \Pr[P_k \in A \mid P_i = \sigma] \). Then the following proposition holds:

**Proposition 6.** For every \( i < k \) and \( A \subseteq X(k) \), it holds that

\[
\| A \| = \sum_{\sigma \in X(i)} w(\sigma) \cdot \| A_{\sigma}\|_{\sigma}.
\]

**Proof.** Fix \( i < k \). By the law of total probability

\[
\|A\| = \Pr[P_k \in A] = \sum_{\sigma \in X(i)} \Pr[P_i = \sigma] \cdot \Pr[P_k \in A \mid P_i = \sigma] = \sum_{\sigma \in X(i)} w(\sigma) \cdot \| A_{\sigma}\|_{\sigma},
\]

which holds since the events \( \{ P_i = \sigma \}_{\sigma \in X(i)} \) are a partition of the sample space. \( \square \)

### 3.2 Formal statement of the main theorem

Let us now formally define what the “nice links” in Theorem 4 are. For every face \( \sigma \) in the complex, we need that \( X_{\sigma} \) has the following structure:

1. \( X_{\sigma} \) is a \( \beta \)-co-boundary expander (as a complex).

2. The graph induced by the vertices \( X_{\sigma}(0) \) and the edges \( X_{\sigma}(1) \) is, loosely speaking, a “good spectral expander graph”. More accurately, we say that a \( d \)-dimensional simplicial complex \( X \) is an \( \alpha \)-spectral expander if \( X(0) \cup X(1) \) is such that for all \( S \subseteq X(0) \), we have \( \|E(S)\| \leq \|S\| (\|S\| + \alpha) \).

We are now ready to formally state the main theorem.

\*Note that we slightly abuse the term “spectral expander” here, since the expansion that we require from \( X(0) \cup X(1) \) is not exactly spectral expansion.
Theorem 7. Let $X$ be a $d$-dimensional simplicial complex with the following properties:

- **Bounded degree:** The degree of every $v \in X(0)$ is at most $Q$.
- **Nice links:**
  - $\forall \ 0 \leq k < d, \forall \sigma \in X(k), \ X_\sigma$ is a $\beta$-co-boundary expander.
  - $\forall \ -1 \leq k < d, \forall \sigma \in X(k), \ X_\sigma$ is an $\alpha$-spectral expander.

Then, $X^{(d-1)} = X \setminus X(d)$ is an $(\varepsilon, \mu)$-co-systolic expander, where $\varepsilon = \varepsilon(Q, \beta, \alpha)$ and $\mu = \mu(\beta, d)$.

Note that we get co-systolic expansion not for the entire complex, but only for the first $d-1$ levels (i.e., without the top level). Also note that the second requirement under “nice links” also requires that the “link” of the empty set $X_\emptyset$, which is the entire complex, would be an $\alpha$-spectral expander. This property can actually be inferred from the fact that all the other links are expanders; but we will not discuss this.

As we mentioned above, the Ramanujan complex is bounded degree. Moreover, all of its links are “nice” in the sense that they admit the properties above, by the following claim:

Claim 8. The spherical building $\mathbb{P}^{d-1}(\mathbb{F}_p)$ satisfies the following:

- The building is a $\beta$-co-boundary expander with a constant $\beta > 0$.
- The building is an $\alpha$-spectral expander with $\alpha \sim \Theta\left(\frac{4}{\sqrt{p}}\right)$.

It follows that the Ramanujan complex is indeed a co-systolic expander. We will not prove Claim 8, but rather focus on proving Theorem 7. Let us now mention the two main steps in the proof of Theorem 7.

Proof Sketch for Theorem 7. The two main steps in the proof are:

1. We would show that small sets expand well (main step).
2. We would show that all sets expand, and here we lose the last level.

In the next lecture we will review the proof in more detail.

References