ENGINEERING MATHEMATICS - TUTORIAL AND HOMEWORK 3

Please submit all exercises below in hard copy (either in English or in Hebrew) next Thursday, November 19^{th}, in my mailbox in the Ziskind (main math) building. Sections marked as bonus are not mandatory.

1. Vector spaces (reminder)

Definition: Let $\mathbb{F}$ be a field. A set V together with two maps $\cdot : \mathbb{F} \times V \to V$ (scalar multiplication) and $+ : V \times V \to V$ (addition) is called a vector space over $\mathbb{F}$ (or a linear space over $\mathbb{F}$) if the following properties hold:

(1) $\forall v, u, w \in V : (v + u) + w = v + (u + w)$ (addition is associative)
(2) $\exists \theta \in V$ such that $\forall v \in V : \theta + v = v + \theta = v$ (existence of a "zero" vector)
(3) $\forall v \in V\ \exists v' \in V$ such that $v + v' = v' + v = \theta$ (existence of "inverse")
(4) $\forall v, u \in V : v + u = u + v$ (addition is commutative)
(5) $\forall v \in V : 1 \cdot v = v$
(6) $\forall \alpha, \beta \in \mathbb{F}$ and $\forall v \in V : \alpha \cdot (\beta \cdot v) = \alpha\beta \cdot v$ (scalar multiplication is associative)
(7) $\forall \alpha, \beta \in \mathbb{F}$ and $\forall v \in V : (\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$ (distributivity of scalars)
(8) $\forall \alpha \in \mathbb{F}$ and $\forall v, u \in V : \alpha \cdot (v + u) = \alpha \cdot v + \alpha \cdot u$ (distributivity of vectors)

Remark: a set V satisfying properties (1)-(3) is called a group. A group satisfying property (4) as well, is called a commutative (or an Abelian) group.

Remark: we often omit the dot indicating the scalar product when it is clear from the context what do we mean. We usually write 0 instead of θ. Note that the "zero" vector of V is not the zero element of the field $\mathbb{F}$.

1.1. Examples: any field is a vector space over itself, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, the space of all real functions (i.e. functions from $\mathbb{R}$ to itself) is a vector space over $\mathbb{R}$, $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Definition: Let V be a vector space over $\mathbb{F}$. A subset $V' \subset V$ is called a linear subspace of V if V' is itself a linear space over $\mathbb{F}$ with the same operations of scalar multiplication and addition.

1.2. Examples: $\mathbb{R}^2 \subset \mathbb{R}^3$ is a linear subspace, all real polynomials (i.e. polynomials with real coefficients) form a linear subspace of the space of all real functions, $\mathbb{R} \subset \mathbb{C}$ is a linear subspace over $\mathbb{R}$ (and not over $\mathbb{C}$).

2. Linear dependence and basis (reminder)

Definition: A (maybe infinite) set of vectors $A \subset V$ is called linearly dependent if there exist $\{\alpha_i\}_{i=1}^n \subset \mathbb{F}$ not all zero, and $\{v_i\}_{i=1}^n \subset A$ such that $\sum_{i=1}^n \alpha_i v_i = 0$. Note that although A may be infinite, n is finite, i.e. we are only allowed to take finitely many vectors in A in the above linear combination. A is called linearly independent otherwise.

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2.1. Exercise: In each of the following cases, is the following set of vectors linearly independent? Check using the definition of linear dependence. If the answer is no, give an explicit linear dependency (i.e. find some scalars $\{\alpha_i\}_{i=1}^n$ not all zero such that $\sum_{i=1}^n \alpha_i v_i = 0$). If yes, prove it.

(1) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

(2) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

(3) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(4) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

(5) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$

(6) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

(7) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$

(8) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$

2.2. Definition: Let $A \subset V$ be some (maybe infinite) set of vectors. We define the span of A by $span\{A\} = \{\sum_{i=1}^n \alpha_i v_i | \alpha_i \in \mathbb{F}, v_i \in A\}$, i.e. all linear combinations of finitely many vectors in A.

Remark: For any set $A \subset V$: $span\{A\} \subset V$ is a linear subspace (make sure you understand why).

Definition: A set of vectors $B \subset V$ is called a basis of V if $span\{B\} = V$ and B is linearly independent.

Recall that a vector space V is called finite dimensional if there exists a basis of V with finitely many vectors. In that case all bases of V are finite, and moreover all of them have the same number of vectors. This number is called the dimension of V and is denoted by $dimV$.

3. Linear maps (reminder)

Definition: Let V, W be two vector spaces over $\mathbb{F}$. A map $f : V \to W$ is called linear if $\forall v_1, v_2 \in V$ and $\forall \alpha \in \mathbb{F}$: $f(v_1 + v_2) = f(v_1) + f(v_2)$ and $f(\alpha v_1) = \alpha f(v_1)$.

Recall the construction of a matrix corresponding to a linear map $f : V \to W$ in a given basis $\{e_i\}_{i=1}^n$ of V and a given basis $\{l_j\}_{j=1}^m$ of W: this is an $n \times m$ matrix whose i's column in the coordinates of $f(e_i)$ in the basis $\{l_j\}_{j=1}^m$. If $V = W$ and we choose the same basis twice (i.e. $\{e_i\}_{i=1}^n = \{l_j\}_{j=1}^m$ with the same order of vectors) we say that the matrix is the corresponding matrix of A with respect to the basis $\{e_i\}_{i=1}^n$.

3.1. Exercise: Let $f : V \to W$ be a linear map. What can you say about $f(0)$? Here 0 is the zero element of the vector space (θ above), e.g $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^2$. Make sure you only use the definition of a linear map, the axioms of a vector space or claims you already proved.

3.2. Exercise: In each of the following cases, determine whether the given map is linear or not. If it is, prove you claim and write out the corresponding matrix with respect to the "standard" basis ($\{x\}$ for $\mathbb{R}$,$\{x, y\}$ for $\mathbb{R}^2$ and $\{x, y, z\}$ for $\mathbb{R}^3$). If not, give an explicit example which shows that the given map is not linear.

(1) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x/5$
(2) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + 2$
(3) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$
(4) $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} x \\ x^3 \end{pmatrix}$
(5) $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x$
(6) $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2x + y$
(7) $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2x + y + 5$
(8) $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto xy$
(9) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x + 2y \end{pmatrix}$
(10) $f : \mathbb{R}^2 \to \mathbb{R}^2$, f is rotation around the origin (point $(0, 0)$) by angle $\frac{\pi}{2}$.
(11) $f : \mathbb{R}^2 \to \mathbb{R}^2$, f is rotation around the point $(1, 0)$ by angle $\frac{\pi}{3}$.
(12) $f : \mathbb{R}^2 \to \mathbb{R}^2$, f is translation by vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
(13) $f : \mathbb{R}^2 \to \mathbb{R}^2$, f is reflection with respect to the line $x = 1$.
(14) $f : \mathbb{R}^2 \to \mathbb{R}^2$, f is reflection with respect to the line $y = x$.
(15) $f : \mathbb{R}^2 \to \mathbb{R}^2$, f is a glide reflection with respect to the line $y = 2x$ and vector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.
(16) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z \\ x \\ y \end{pmatrix}$

3.3. Example: (choosing a basis) The map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x) = 2x, f(y) = 0$ takes a point in the plain, projects it to the x axis, and then doubles its length. In the "standard" basis $\{x, y\}$ it is given by the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. The map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(x) = 0, g(y) = 2y$ takes a point in the plain, project it to the y axis, and then doubles its length. In the "standard" basis $\{x, y\}$ it is given by the matrix $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Clearly the matrices A and B are not the same, and indeed if we think of them as representing linear maps in the same basis, these maps are different. One easily sees that turning your head counter clockwise by $\pi/2$, f "becomes" g, i.e. if we would have chosen x to be y and y to be $-x$ we would get g instead of f. Indeed, if we write the corresponding matrix to map g with respect to the basis $\{y, -x\}$ we get the matrix A. Such two matrices are called similar.

3.4. Exercise: The linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given with respect to the basis $\{x, y\}$ by the matrix $\begin{pmatrix} \frac{7}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{7}{2} \end{pmatrix}$. The linear map $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given with respect to the basis $\{x, y\}$ by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$. Write the corresponding matrix to f with respect to the basis $\{x + y, x - y\}$ and conclude that the matrices $\begin{pmatrix} \frac{7}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{7}{2} \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ are similar.

3.5. Exercise: (bonus) Let $f : V \to W$ be a linear map. Define the kernel of f: $Ker(f) = \{v \in V | f(v) = 0\}$ and the image of f: $Im(f) = \{w \in W | \exists v \in V$ such that $f(v) = w\}$. Show that both $Ker(f)$ and $Im(f)$ are closed with respect to addition and scalar multiplication, i.e. $\forall v, w \in ker(f)$ also $v + w \in ker(f)$, and $\forall v \in ker(f)$ and $\forall \alpha \in \mathbb{F}$ also $\alpha v \in ker(f)$, and similarly for the image. We conclude that the kernel (respectively the image) of any linear map is a linear subspace of V (resp. W).

3.6. Exercise: (bonus) Let $\mathbb{R}[x]$ be the vector space of all real polynomials in one variable. Which of the following maps are linear?

(1) $f : \mathbb{R}[x] \to \mathbb{R}$ given by substituting $x = 0$ (i.e. for any polynomial $P(x)$: $f(P(x)) = P(0)$).
(2) $f : \mathbb{R}[x] \to \mathbb{R}$ given by substituting $x = 1$ (i.e. for any polynomial $P(x)$: $f(P(x)) = P(1)$).
(3) $f : \mathbb{R}[x] \to \mathbb{R}[x]$ given by derivation (i.e. for any polynomial $P(x) : f(P(x)) = P'(x) = \frac{dP(x)}{dx}$).
(4) $f : \mathbb{R}[x] \to \mathbb{R}[x]$ given by squaring (i.e. for any polynomial $P(x) : f(P(x)) = P^2(x)$).
(5) $f : \mathbb{R}[x] \to \mathbb{R}[x]$ given by adding 1 to the argument (i.e. for any polynomial $P(x) : f(P(x)) = P(x + 1)$).

4. Maps composition and matrix multiplication (reminder)

4.1. Exercise: Let V, W, U be three finite dimensional vector space of a field $\mathbb{F}$. Let $f : V \to W, g : W \to U$ be two linear maps. Prove using the definition of a linear map that $g \circ f : V \to U$ is also a linear map.

4.2. Definition: Let A be an $n \times m$ matrix, B be an $m \times l$ matrix. We define the multiplication $C = A \cdot B$ to be an $n \times l$ matrix whose entries are given by $(A \cdot B)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$.

4.3. Claim: Let V, W, U be three finite dimensional vector space of a field $\mathbb{F}$. Let A be the matrix corresponding to to the linear map $f : V \to W$ with respect to the bases $\{v_i\}_{i=1}^n$ of V and $\{w_i\}_{i=1}^m$ of W. Let B be the matrix corresponding to to the linear map $g : W \to U$ with respect to the bases $\{w_i\}_{i=1}^m$ of W and $\{u_i\}_{i=1}^l$ of U. Then the matrix corresponding to to the linear map $g \circ f : V \to U$ with respect to the bases $\{v_i\}_{i=1}^n$ of V and $\{u_i\}_{i=1}^l$ of U is $B \cdot A$.

Remark: the proof of the claim is very technical and straight forward.

4.4. Exercise: Prove that matrix multiplication is associative (hint: you can either use definition 4.2 or claim 4.3).

4.5. Exercise: Give an explicit formula (without matrices) for the following compositions $g \circ f$ of linear maps f, g. After that, write out the matrices corresponding to each of the linear maps $f, g, g \circ f$ and check that the multiplication of matrices of g and f is indeed the matrix of $g \circ f$.

(1) $f : \mathbb{R} \to \mathbb{R}^3$, $x \mapsto \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix}$,

$g : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+z \\ z+2y \\ y+x \end{pmatrix}$

(2) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ x+y+z \end{pmatrix}$,

$g : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+2y \\ 2x-y \end{pmatrix}$

Recall that the matrix corresponds to rotation around the point $(0,0)$ (the origin) by the angle α with respect to the "standard" basis $\{x, y\}$ is $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

(3) f is rotation (counterclockwise) around the origin by the angle α, g is rotation around the origin by the angle β (write out the matrices using α and β).

(4) f is rotation around the the origin by angle α, g is reflection with respect to the line $y = 0$ (write out the matrices using α).

5. Invertibility (reminder)

The matrix corresponds to the identity map from an n-dimensional vector space to itself (with respect to any basis one fixes) is the $n \times n$ matric with 1's on the main diagonal and zeros anywhere else (e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in $\mathbb{R}^3$). It is called the identity matrix, and denoted by $Id_{n \times n}$

Definition: We call a linear map $f : V \to W$ invertible if there exists a linear map $g : W \to V$ such that $f \circ g$ is the identity map on W and $g \circ f$ is the identity map on V.

Let us quote three useful theorems (without proving them):

5.1. Theorem: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is an invertible linear map then $n = m$.

5.2. Theorem: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a bijective (i.e. one to one and onto) linear map then $n = m$ and f is invertible.

5.3. Theorem: If $f : \mathbb{R}^n \to \mathbb{R}^n$ is either one to one or onto linear map, then it is both one to one and onto (and so invertible by 5.2).

Definition: We call an $n \times m$ matrix A invertible if it corresponds (with respect to some basis) to an invertible linear map $f : \mathbb{R}^n \to \mathbb{R}^m$.

By 5.1 if A is an invertible $n \times m$ matrix then $n = m$, and it is easy to see that A is invertible if and only if there exists an $n \times n$ matrix B such that $A \cdot B = B \cdot A = Id_{n\times n}$ (make sure you understand why). Such B is called the inverse of A, and we denote $A^{-1} := B$.

5.4. Exercise: Let A be an invertible matrix. Prove that the inverse of A is uniquely defined and thus the definition of "the" inverse makes sense, i.e. if both B_1 satisfy $A \cdot B_1 = B_1 \cdot A = Id_{n\times n}$ and B_2 satisfy $A \cdot B_2 = B_2 \cdot A = Id_{n\times n}$ then $B_1 = B_2$ (hint: first prove that for any $n \times n$ matrix M we have: $Id_{n\times n} \cdot M = M \cdot Id_{n\times n} = M$, and then use exercise 4.4).

5.5. Exercise: For each of the linear maps you found in 3.2 check whether the map is invertible. If not, prove your claim (you can use the theorems above). If yes, describe the inverse map f^{-1} (without using matrices), and write out the inverse of the matrix corresponding to f (i.e. if f corresponds to matrix A, write out A^{-1} explicitly).

5.6. Exercise: (bonus) For each of the linear maps you found in 3.6 check whether the map is invertible. If not, prove your claim. If yes, describe the inverse map f^{-1}. Note that as $\mathbb{R}[x]$ is not finite dimensional you can neither use matrix representation nor the theorems above.

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