ENGINEERING MATHEMATICS - TUTORIAL AND HOMEWORK 6

Please submit all exercises below in hard copy (either in English or in Hebrew) on Thursday two weeks from today, December 17^{th}, in the tutorial. Sections marked as bonus are not mandatory.

1. Directed Volumes in $\mathbb{R}^n$

1.1. Let $\{e_1, e_2, e_3, ..., e_n\}$ be some basis of $\mathbb{R}^n$, for instance $\{x, y, z\}$ in $\mathbb{R}^n$. We want to define a volume function:

$$Vol : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{n times}} \to \mathbb{R}.$$

This function should take n vectors in $\mathbb{R}^n$ and tell us "what is the volume caught inside the parallelepiped (MAKBILON) they form". What do we want from such a function?

First, we want to have some scale. In order to do so we demand that:

$$(1)\ Vol(e_1, e_2, e_3, ..., e_n) = 1.$$

Second, we want to have "linearity in each vector", e.g. if we double the length of any vector we double the volume of our MAKBILON and if we add a vector in one place and leave all others untouched we get the sum of volumes.

In order to do so we demand that for any set of vectors $a_1, a_2, ..., a_n, v \in \mathbb{R}^n$ and for any scalar $\alpha \in \mathbb{R}$:

$$\begin{gathered}(2)\ Vol(x_1, x_2, x_3, ..., x_{j-1}, x_j + \alpha v, x_{j+1}, x_{j+2}, ..., x_n) = \\ Vol(x_1, x_2, x_3, ..., x_{j-1}, x_j, x_{j+1}, x_{j+2}, ..., x_n) + \\ \alpha Vol(x_1, x_2, x_3, ...x_{j-1}, v, x_{j+1}, x_{j+2}, ..., x_n).\end{gathered}$$

Draw yourself a picture in $\mathbb{R}^2$ and calculate the 2 dimensional volume (area) of the parallelogram defined by $\{x, x + 2y\}$. Now calculate the areas of the two parallelograms defined by $\{x, y\}$ and by $\{x, x + y\}$. If you made it right you will see that the sum of areas of these two is exactly the area of the first.

Third, and this might seems to be a bit less natural, we want the volume function to be directed, that is to have sign (i.e. sometimes the volume is negative). It can be shown from the first and the second demands, when we take α to be minus one: $Vol(-e_1, e_2, e_3, ..., e_n) = -Vol(e_1, e_2, e_3, ..., e_n) = -1$.

Let us explain this in $\mathbb{R}^2$ using non-mathematical hand waiving, and then give the general formalism: if we have two vectors $v_1, v_2 \in \mathbb{R}^2$ we construct the corresponding parallelogram by walking along v_1 first, then continue to walk along the translated v_2, then along the translated $-v_1$ and finally along the translated $-v_2$. The corresponding parallelogram to $v_2, v_1 \in \mathbb{R}^2$ would be constructed by walking along v_2 first, then continue

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to walk along the translated v_1, then along the translated $-v_2$ and finally along the translated $-v_1$. These two parallelograms are almost the same, but they have different "orientation". Take v_1 to be x and v_2 to be y you will get that the first was constructed by walking counterclockwise, and the second by walking clockwise. Thus we want to say that they have the same area, but with different signs.

The general demand would be that:

$$(3)\ Vol(x_1, x_2, x_3, ..., x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, ..., x_n) =$$
$$-Vol(x_1, x_2, x_3, ..., x_{j-2}, x_j, x_{j-1}, x_{j+1}, x_{j+2}, ..., x_n)$$

i.e. switching two vectors (it is enough to demand for adjacent) gives a minus sign.

1.2. Theorem: for any basis $\{e_1, e_2, e_3, ..., e_n\}$ of $\mathbb{R}^n$ there exists a unique volume function $Vol : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{n times}} \to \mathbb{R}$ satisfying (1),(2) and (3) above.

2. How a linear transformation blows up space

2.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. We want to ask "how does this f blows up (or shrinks) $\mathbb{R}^n$"?

We fix some basis $\{e_1, e_2, e_3, ..., e_n\}$ of $\mathbb{R}^n$, so by Theorem 1.2 above we have a volume function on $\mathbb{R}^n$, that satisfy $Vol(e_1, e_2, e_3, ..., e_n) = 1$. So we may ask what is the volume of the image of this parallelepiped under f, i.e. what is $Vol(f(e_1), f(e_2), f(e_3), ..., f(e_n))$.

Note that if we would have taken a different basis $\{e'_1, e'_2, e'_3, ..., e'_n\}$ of $\mathbb{R}^n$ we would have gotten a different volume function - in order to stress it we will call this function Vol'. In that case we would have $Vol'(e'_1, e'_2, e'_3, ..., e'_n) = 1$, and we would ask what is $Vol'(f(e'_1), f(e'_2), f(e'_3), ..., f(e'_n))$. The following remarkable Theorem says that these two numbers are the same!

2.2. Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and $\{e_1, e_2, e_3, ..., e_n\}$,$\{e'_1, e'_2, e'_3, ..., e'_n\}$ two bases of $\mathbb{R}^n$. Let Vol, Vol' be the two volume function corresponding to these bases. Then $Vol(f(e_1), f(e_2), f(e_3), ..., f(e_n)) = Vol'(f(e'_1), f(e'_2), f(e'_3), ..., f(e'_n))$.

2.3. Definition: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, $\{e_1, e_2, e_3, ..., e_n\}$ a basis of $\mathbb{R}^n$ and Vol the corresponding volume function to this basis. We define the determinant of f by:

$$Det(f) := Vol(f(e_1), f(e_2), f(e_3), ..., f(e_n)),$$

and by Theorem 2.2 above this does not depend on the basis chosen.

2.4. Example: let us calculate the determinant of the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(e_1) = 8e_1$ and $f(e_2) = 15e_1 + e_2$ (where $\{e_1, e_2\}$ is some basis of $\mathbb{R}^2$). We use two facts:

(1) $Vol(e_1, e_2) = 1$ (by the first property of the volume function).

(2) $Vol(e_1, e_1) = -Vol(e_1, e_1)$ (by the third property of the volume function), so we must have that $Vol(e_1, e_1) = 0$.

Now it is easy to compute using the second property of the volume function:

$$Det(f) = Vol(f(e_1), f(e_2)) = Vol(8e_1, 15e_1 + e_2) =$$

$$120 \cdot Vol(e_1, e_1) + 8 \cdot Vol(e_1, e_2) = 120 \cdot 0 + 8 \cdot 1 = 8.$$

2.5. Exercise: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Assume there exists a non-zero vector $v \in \mathbb{R}^n$ (i.e. v is not the neutral element, which we also call the origin) such that $f(v) = 0$ (i.e. $f(v)$ is the origin). Prove that $Det(f) = 0$ (hint: you may use the fact that a linearly independent set in $\mathbb{R}^n$ can always be completed to a basis, and then use the fact that the volume function is linear).

2.6. Exercise: let $v_1, v_2, ..., v_n \in \mathbb{R}^n$. Prove that if there exist $i \neq j$ such that $v_i = v_j$ then $Vol(v_1, v_2, ..., v_n) = 0$ (hint: this is very easy).

Remark: I remind you that you have to solve all exercises only using things we already proved or quoted in this course. Please do not use any theorems you saw elsewhere. In particular note that for exercises 2.5 and 2.6 you are not allowed to use any matrices whatsoever.

3. Matrix representation

Now that we defined the notion of a determinant of a linear function, we want to develop a tool to calculate it. This tool would be the determinant of a matrix.

3.1. Definition: let us denote by $Mat_{n\times n}(\mathbb{R})$ the set of all $n\times n$ matrices with real entries. Given $A \in Mat_{n\times n}(\mathbb{R})$ we denote its entries by a_{ij} (i.e. a_{ij} is the element of the matrix A standing in the i's row and j's column). We define a function $Det : Mat_{n\times n}(\mathbb{R}) \to \mathbb{R}$ that we call the determinant of A by:

$$Det(A) := \Sigma_{\sigma\in S_n} sgn(\sigma)\Pi_{i=1}^{n} a_{i\sigma(i)}.$$

3.2. Example: if A is a 2×2 matrix then $Det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

What is the connection between the determinant of a linear function and the determinant of a matrix? Well, we have a geometrical intuition and motivation for the first (the determinant of a function tells us what does this function do to volumes), while the second seems out of the blue. The answer is the following Theorem:

3.3. Theorem: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and let $\{e_1, e_2, e_3, ..., e_n\}$ be a basis of $\mathbb{R}^n$. Denote by A the matrix corresponding to f with respect to the basis $\{e_1, e_2, e_3, ..., e_n\}$ (see section 3 of tutorial 3 regarding how to construct this matrix). Then $Det(f) = Det(A)$.

Now we can easily conclude:

3.4. Corollary: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and let $\{e_1, e_2, e_3, ..., e_n\}$,$\{e'_1, e'_2, e'_3, ..., e'_n\}$ be two bases of $\mathbb{R}^n$. Denote by A the matrix corresponding to f with respect to the basis $\{e_1, e_2, e_3, ..., e_n\}$ and by A' the matrix corresponding to f with respect to the basis $\{e'_1, e'_2, e'_3, ..., e'_n\}$. Then $Det(A) = Det(A')$.

3.5. Example: the matrix of f defined in 2.4 with respect to the basis $\{e_1, e_2\}$ above is $\begin{pmatrix} 8 & 0 \\ 15 & 1 \end{pmatrix}$. By example 3.2 we have $Det \begin{pmatrix} 8 & 0 \\ 15 & 1 \end{pmatrix} = 8 \cdot 1 - 0 \cdot 15 = 8$, which is exactly $Det(f)$ as we calculated in 2.4.

3.6. Exercise: let $\{x, y\}$ be some basis of $\mathbb{R}^2$. Define a linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = x+3y$, $f(y) = 5x+7y$. calculate the determinant of f in four different ways (they will all give you the same number of course, which is -8):

(1) calculate directly the volume of the parallelogram formed by $f(x)$ and $f(y)$ (like we did in 2.4). Do not use any martices!
(2) construct the matrix of f with respect to the basis $\{x, y\}$ and calculate its determinant using example 3.2.
(3) construct the matrix of f with respect to the basis $\{x+y, x-y\}$ and calculate its determinant using example 3.2.
(4) construct the matrix of f with respect to the basis $\{x + \frac{3+2\sqrt{6}}{5}y, x + \frac{3-2\sqrt{6}}{5}y\}$ and calculate its determinant using example 3.2.

3.7. Remark: if you solved the exercise correctly, you noticed that the fourth way, though at first seems much more complicated, brought you to a matrix whose only non-zero entries sit on its main diagonal. Such matrix is called a diagonal matrix, and it is very easy to calculate the determinant of such matrices (not only in the 2×2 case). Intuitively you may think of the basis chosen in (4) as a "natural" basis for this f - in that special basis all f does is to multiply the first basis vector by one number, and the second by another.

In the following two exercises you may use the fact that $\forall v \in \mathbb{R}^n$: $0 \cdot v$ is the neutral element of $\mathbb{R}^n$.

3.8. Exercise: let $A \in Mat_{n\times n}(\mathbb{R})$. Assume A has a column constructed of zeroes only. Prove that $Det(A) = 0$ in two different ways:

(1) use definition 3.1.
(2) construct a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\{e_1, e_2, ..., e_n\}$ of $\mathbb{R}^n$ such that A is the matrix corresponding to f with respect to this basis. Then use exercise 2.5.

3.9. Exercise: let $A \in Mat_{n\times n}(\mathbb{R})$. Assume A has two equal columns (i.e. there exist $j \neq j'$ such that for any i: $a_{ij} = a_{ij'}$). prove that $Det(A) = 0$ (hint: use exercise 2.6).

4. Some geometric intuition in $\mathbb{R}^2$

We want to verify some geometric facts we "feel" should be true. For instance, a rotation around the origin (in any angle) does not change volumes at all - so we would expect it to have determinant 1. A reflection (with respect to any line passing through the origin) switches the orientation of any parallelogram (clockwise to counterclockwise and vise versa) but does not blow it up or shrink it - so we would expect it to have determinant -1. Next week we will define the notion of distance and call linear functions that preserve distances Orthogonal maps. We will prove that they all have determinant ± 1. Beware, not all linear functions that have determinant ± 1 preserves distances, e.g. $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ (given in any basis) in $\mathbb{R}^2$ has determinant 1, but it does not preserve distances.

4.1. Exercise: for each of the following linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ write the corresponding matrix with respect to a basis of your choice, and compute its determinant.

(1) a rotation around the origin in angle α counterclockwise.
(2) a reflection with respect to the line $ax + by = 0$ (a and b are two real numbers, where at least one of them is not zero).
(3) the linear map defined by $v \mapsto 2v$ (i.e. the linear map that takes any vector in $\mathbb{R}^2$ and doubles its length).

4.2. Exercise: (bonus) Why do we always only deal with rotations around the origin in $\mathbb{R}^2$ counter clockwise? Why do we leave out the rotations clockwise?

4.3. Exercise: Solve Problem 13 on p. 114 in the book "Math and Technology".

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