ENGINEERING MATHEMATICS - TUTORIAL AND HOMEWORK 7

Please submit all exercises below in hard copy (either in English or in Hebrew) next Thursday, December 24^{th}, in the tutorial. Sections marked as bonus are not mandatory.

1. Inner products in $\mathbb{R}^n$

Let $\{e_1, e_2, e_3, ..., e_n\}$ be some basis of $\mathbb{R}^n$. We want to define a function that will enable us to say that the length of each e_i is 1, and to say that each two distinct basis vectors are orthogonal.

1.1. Definition: let $\{e_1, e_2, e_3, ..., e_n\}$ be some basis of $\mathbb{R}^n$. We define a function

$$< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

by:

$$< v, w >= \Sigma_{i=1}^n v_i w_i,$$

where v_i (respectively w_i) is the i^{th} coordinate of v (resp. w) in the basis $\{e_1, e_2, e_3, ..., e_n\}$ (i.e. $v = \Sigma_{i=1}^n v_i e_i$, $w = \Sigma_{i=1}^n w_i e_i$). We call this function the inner product on $\mathbb{R}^n$ with respect to the basis $\{e_1, e_2, ..., e_n\}$, and $< v, w >$ is called the inner product of v and w.

1.2. Definition: we call two vectors in $v, w \in \mathbb{R}^n$ orthogonal if $< v, w >= 0$.

1.3. Exercise: let $\{e_1, e_2, e_3, ..., e_n\}$ be some basis of $\mathbb{R}^n$ and $< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the corresponding inner product. Prove the following properties:

(1) inner product is linear in all variables, i.e. for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and any $v_1, v_2, w_1, w_2 \in \mathbb{R}^n$:

$$< \alpha_1 v_1 + \alpha_2 v_2, \beta_1 w_1 + \beta_2 w_2 >=$$

$$\alpha_1\beta_1 < v_1, w_1 > +\alpha_1\beta_2 < v_1, w_2 > +\alpha_2\beta_1 < v_2, w_1 > +\alpha_2\beta_2 < v_2, w_2 > .$$

(2) inner product is symmetric, i.e. for any $v, w \in \mathbb{R}^n$: $< v, w >=< w, v >$.

(3) for any $v \in \mathbb{R}^n$: $< v, v >\geq 0$ and moreover $< v, v >= 0$ if and only if $v = 0$ (you may use the fact that the coordinates of the zero element of $\mathbb{R}^n$ in any basis are all zeroes).

(4) for any $v \in \mathbb{R}^n$: $< v, e_i >= v_i$ (where v_i is the i^{th} coordinate of v in the basis $\{e_1, e_2, e_3, ..., e_n\}$), i.e. $v = \Sigma_{i=1}^n < v, e_i > e_i$.

Motivated by section (3) in exercise 1.3 we give the following definitions:

1.4. Definition: given an inner product on $\mathbb{R}^n$ we define a function $| \cdot | : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by $|v| = \sqrt{< v, v >}$. We call $|v|$ the length (or the norm) of v.

1.5. Definition: given an inner product on $\mathbb{R}^n$ we define a function $dist : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by $dist(v, w) = |v - w|$. We call this function the distance function, and using section (1) of exercise 1.3 it is easy to see that it is symmetric (i.e. $dist(v, w) = dist(w, v)$).

It is important to stress that all the above notions (inner product, orthogonality, length and distance) are strongly dependent on the basis chosen. The following exercise illustrates this fact.

Date: December 17^{th} 2015.

1.6. Exercise: let $v, w \in \mathbb{R}^2$ be two (non-zero) linearly independent vectors. In each of the following bases of $\mathbb{R}^2$ answer the following questions: what is the norm of v? What is the norm of w? Are v and w orthogonal? What is the distance between v and w? What is the inner product of v and w?

(1) $\{v, w\}$.
(2) $\{2v, w\}$.
(3) $\{v + w, v - w\}$.
(4) $\{v, v + w\}$.

2. Geometric intuition (angles)

2.1. Exercise: let $\{e_1, e_2\}$ be some basis of $\mathbb{R}^2$ and $< \cdot, \cdot >$ its corresponding inner product. Prove that for any two non-zero $v, w \in \mathbb{R}^2$:

$$\left| \frac{< v, w >}{|v| \cdot |w|} \right| \leq 1.$$

Hint: it is enough to show that $(\frac{<v,w>}{|v| \cdot |w|})^2 \leq 1$ or that $(|v| \cdot |w|)^2 - (< v, w >)^2 \geq 1$.

2.2. Definition: by exercise 2.1 for any two non-zero vectors v, w in $\mathbb{R}^2$ there exists a unique real number $\theta \in [0, \pi]$ such that $cos(\theta) = \frac{<v,w>}{|v| \cdot |w|}$. We call this number the angle between v and w.

Now we have some geometric intuition for the inner product in $\mathbb{R}^2$: it can now we written as $< v, w >= |v| \cdot |w| \cdot cos(\theta)$. We note that $|v| cos(\theta)$ can be realized as the length of the projection of v on the line (one dimensional space) spanned by w, and vise versa $|w| cos(\theta)$ can be realized as the length of the projection of w on the line spanned by v.

This intuition is true for $\mathbb{R}^n$ in general. The inequality you proved in exercise 2.1 also holds there, and the angle θ can be visualized as the angke between v and w in the plain (two dimensional space) spanned by both.

2.3. Exercise: let us show that the notion of an angle strongly depends on the basis chosen: what is the angle between the two vectors v and w of exercise 1.6 in each basis (1),(2),(3) and (4)?

3. Transposed matrix

3.1. Definition: let $A \in Mat_{n \times n}(\mathbb{R})$. We define the transposed matrix $A^t \in Mat_{n \times n}(\mathbb{R})$ by $(A^t)_{ij} = A_{ji}$ (i.e. we "reflect" the matrix with respect to its main diagonal such that its columns become its rows and vise versa).

3.2. Exercise: take some linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ and fix a basis $\{e_1, e_2, e_3, ..., e_n\}$ of $\mathbb{R}^n$. We now have the notion of an inner product and also a matrix $A \in Mat_{n \times n}(\mathbb{R})$ that corresponds to f with respect to this basis. Prove that for any $v, w \in \mathbb{R}^n$:

$$< Av, w >=< v, A^t w > .$$

Notation: after choosing a basis of $\mathbb{R}^n$ we can write any vector v as a column $\begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{pmatrix}$,

and then we can calculate

$$< v, w >=< \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{pmatrix} >= \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{pmatrix}^t \cdot \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & . & . & . & v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{pmatrix},$$

where in the last two expressions we have matrix multiplication.

4. Distance preserving linear maps

We fix a basis of $\mathbb{R}^n$.

4.1. Definition: We call a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ distance preserving (or Orthogonal) if for any two vectors $v, w \in \mathbb{R}^n$: $dist(v, w) = dist(f(v), f(w))$.

4.2. Definition: We call a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ norm preserving if for any vector $v \in \mathbb{R}^n$: $|v| = |f(v)|$.

4.3. Exercise: prove that a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is distance preserving if and only if it is norm preserving (the "if" part is easy, for the "only if" part use the fact that $|v| = dist(v, 0)$ and exercise 3.1 of homework 3).

4.4. Lemma: if f is distance preserving with respect to some basis of $\mathbb{R}^n$ then it is distance preserving with respect to any basis of $\mathbb{R}^n$.

Remark: we will not prove this lemma in this course, however you may use it.

4.5. Theorem: a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is norm preserving if and only if for any two vectors $v, w \in \mathbb{R}^n$: $< v, w >=< f(v), f(w) >$ (i.e. f is inner product preserving). By the above Lemma it is independent of the basis chosen.

Proof: assume f is inner product preserving, then $|f(v)| = \sqrt{< f(v), f(v) >} = \sqrt{< v, v >} = |v|$, thus f is norm preserving.

Assume f is norm preserving. Then on the one hand

$$|f(v + w)|^2 = |v + w|^2 =$$
$$< v + w, v + w >^2 = |v|^2 + |w|^2 + 2 < v, w >=$$
$$|f(v)|^2 + |f(w)|^2 + 2 < v, w > .$$

On the other hand

$$|f(v + w)|^2 =< f(v + w), f(v + w) >=$$
$$< f(v) + f(w), f(v) + f(w) >=$$
$$< f(v), f(v) > + < f(w), f(w) > +2 < f(v), f(w) >=$$
$$|f(v)|^2 + |f(w)|^2 + 2 < f(v), f(w) > .$$

From the equality of these two expressions it follows that $< v, w >=< f(v), f(w) >$. $\square$

4.6. Exercise: prove that an orthogonal map preserves angles between vectors, i.e. for any orthogonal map f and for any two non-zero vectors v, w: the angle between v and w equals the angle between $f(v)$ and $f(w)$ (hint: this is a one line proof).

4.7. Theorem: let f be a linear map, $\{e_1, e_2, ..., e_n\}$ some basis of $\mathbb{R}^n$ and A the corresponding matrix of f. Then f is norm preserving if and only if $A^tA = Id_{n\times n}$.

4.8. Exercise: (bonus) prove Theorem 4.7. Instructions: use Theorem 4.5 and exercise 3.2. Then the "if" part is very easy. For the "only if" part compute $< e_i, v >=< Ae_i, Av >$ and use exercise 1.3.

4.9. We quote two Theorems without giving their proofs:

(1) For any $A, B \in Mat_{n\times n}(\mathbb{R})$:$Det(AB) = Det(A)Det(B)$ (Det is multiplicative).

(2) For any $A \in Mat_{n\times n}(\mathbb{R})$: $Det(A^t) = Det(A)$.

4.10. Corollary: if $A^tA = Id_{n\times n}$ then $Det(A) = \pm 1$.

Proof: $1 = Det(Id_{n\times n}) = Det(A^tA) = Det(A^t)Det(A) = Det(A)^2$.

Remark: Intuitively this make sense: a map that preserves distances should not blow up space or shrink it, so it should have determinant ± 1. An orthogonal map with determinant 1 (respectively -1) is called orientation preserving (resp. orientation changing).

4.11. Exercise: fix some basis of $\mathbb{R}^n$. By doing so we have an inner product $< \cdot, \cdot >$. Fix some non-zero $\alpha \in \mathbb{R}^n$. Consider the map $r_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined by $r_\alpha(\beta) = \beta - \frac{<\alpha,\beta>}{<\alpha,\alpha>}\alpha$.

(1) prove that r_α is linear.
(2) prove that r_α is orthogonal.
(3) prove that $r_\alpha \circ r_\alpha = Id$.
(4) prove that if α and β are orthogonal then $r_\alpha(\beta) = \beta$.
(5) prove that if α and β are proportional (i.e. there exists some $k \in \mathbb{R}$ such that $\beta = k\alpha$) then $r_\alpha(\beta) = -\beta$.
(6) (bonus) interpret r_α geometrically when $n = 2$ and $n = 3$ (i.e. in $\mathbb{R}^2$ and $\mathbb{R}^3$).

5. Orthogonal maps in $\mathbb{R}^2$

You showed in exercise 4.1 of homework 6 that rotation around the origin has determinant 1 and that reflection with respect to any line passing through the origin has determinant -1. We can easily see that these are indeed orthogonal maps as they preserve norm (make sure you understand why!). Let us show that any orthogonal map in the plain is either a rotation or a reflection.

Choose some basis $\{e_1, e_2\}$ of $\mathbb{R}^2$. Assume f is an orthogonal map and that it is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Theorem 4.7 we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and so

$$\begin{aligned} a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1. \\ ab + cd &= 0 \end{aligned}$$

We can find a (unique) real number $0 \leq \alpha < 2\pi$ such that:

$$a = cos(\alpha), c = sin(\alpha), b = \pm sin(\alpha), d = \mp cos(\alpha),$$

where the plus minus signs should be taken respectively.

Now we have two options, either $Det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ or $Det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -1$.

The first will correspond to choosing $b = -sin(\alpha)$, $d = a$ and so we have $\begin{pmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{pmatrix}$, i.e. f is rotation around the origin in angle α counterclockwise.

In the second case we will get $b = sin(\alpha)$, $d = -a$ and so we have $\begin{pmatrix} cos(\alpha) & sin(\alpha) \\ sin(\alpha) & -cos(\alpha) \end{pmatrix}$, i.e. f is reflection with respect to the line given by the equation $-sin(\frac{\alpha}{2})e_1 + cos(\frac{\alpha}{2})e_2 = 0$ (the line spanned by the vector $cos(\frac{\alpha}{2})e_1 + sin(\frac{\alpha}{2})e_2$).

5.1. Exercise: consider $\mathbb{C}$ as a two dimensional vector space over $\mathbb{R}$, and let $\alpha \in \mathbb{C}$. Define a map $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \alpha z$ (i.e. f is the map that takes any complex number and multiplies it by α).

1. prove that f is linear (you may use the fact that multiplying complex numbers is a linear operation).
2. calculate $Det(f)$.
3. for what α's is f an orthogonal map? Among these α's when is f orientation preserving and when it is not?
4. for the cases when f is orthogonal - interpret f geometrically in the complex plain (i.e. explain in a short sentence what does f do geometrically).

6. A remark about Euclidean geometry

In previous weeks we did some exercises about what we called "reflections", mainly in $\mathbb{R}^2$. In fact we were cheating a bit - in order to define what is a reflection with respect to a given line we must have a notion of distance. We always had this notion in mind, though we never defined it precisely. For example when we said "reflection with respect to the line $x = y$" we had in mind a basis $\{x, y\}$, with its inner product, norm, distance and angle functions defined above. Similar idea holds for what we called "rotations".

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