ENGINEERING MATHEMATICS - TUTORIAL AND HOMEWORK 8

Please submit all exercises below in hard copy (either in English or in Hebrew) next Thursday, December 31^{st}, in the tutorial. Sections marked as bonus are not mandatory.

1. Condition for invertibility

1.1. Theorem: let $A \in Mat_{n\times n}(\mathbb{R})$. Then A is invertible if and only if $Det(A) \neq 0$.

1.2. Exercise: prove the "only if" part of Theorem 1.1, i.e. prove that if A is invertible then $Det(A) \neq 0$ (hint: use Theorem 4.9(1) of homework 7).

Remarks: (1) We will not prove the "if" part of Theorem 1.1 in this course, however you may use it. (2) Theorem 1.1 implies that a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $Det(f) \neq 0$ (make sure you understand why).

2. Eigenvalues and Eigenvectors

2.1. Definition: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and let $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$. We say that v is an eigenvector of f with eigenvalue λ if $f(v) = \lambda v$ and $v \neq 0$. We also say that λ is an eigenvalue of f if there exists $v \in \mathbb{R}^n, v \neq 0$ such that $f(v) = \lambda v$. The set of all vectors $v \in \mathbb{R}^n$ that satisfy $f(v) = \lambda v$ is called the eigenspace of λ.

2.2. Exercise: fix a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$. Prove that for any $\lambda \in \mathbb{R}$: the eigenspace of λ is a linear subspace of $\mathbb{R}^n$, and calculate the intersection of the eigenspace of λ_1 and the eigenspace of λ_2 if $\lambda_1 \neq \lambda_2$.

2.3. Theorem: λ is an eigenvalue of f if and only if $Det(\lambda Id - f) = 0$ (where Id is the identity map from $\mathbb{R}^n$ to $\mathbb{R}^n$).

Proof: (λ is an eigenvalue of f) $\iff$ (there exists a non zero $v \in \mathbb{R}^n$ such that $f(v) = \lambda v$) $\iff$ (there exists a non zero $v \in \mathbb{R}^n$ such that $\lambda v - f(v) = (\lambda Id - f)(v) = 0$) $\iff$ ($\lambda Id - f$ is not one to one) $\iff$ ($\lambda Id - f$ is not invertible) $\iff$ ($Det(\lambda Id - f) \neq 0$). The last $\iff$ sign is due to remark (2) above, the one before it is due to Theorems 5.2 and 5.3 of homework 3. Make sure you understand all others. □

Remark: let A be a matrix corresponding to f with respect to some basis. Now it should be clear to you that λ is an eigenvalue of f if and only if $Det(\lambda Id_{n\times n} - A) = 0$.

2.4. Definition: let f be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and A be an $n \times n$ matrix. The polynomial $P_f(x) := Det(xf - A)$ (resp. $P_A(x) := Det(xI_{n\times n} - A)$) is called the characteristic polynomial of the map f (matrix A).

Remark: like in the case of determinant there is a Theorem saying that if A and B are similar matrices (i.e. there exists a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that both A and B represent f in some bases) then $P_A(x) = P_B(x)$. This Theorem does not follow from the one about determinants, but it is also true. Similarly to the determinant case it is also true that $P_f(x) = P_A(x)$, when A is a matrix corresponding to f in some basis.

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2.5. Claim: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then f has no more than n eigenvalues.

Proof: $\lambda \in \mathbb{R}$ is an eigenvalue of $f \iff \lambda \in \mathbb{R}$ is a root of $P_f(x)$ (i.e. $P_f(\lambda) = 0$ and λ is real). As $P_f(x)$ is a polynomial of degree n, by the fundamental theorem of algebra it has no more than n roots. $\square$

So we now can find find eigenvalues of any linear map. We first write the corresponding matrix to f in some basis (let us call this matrix A) and then we find the roots of its characteristic polynomial. Finding eigenvectors is also not difficult: if λ is an eigenvalue then the corresponding eigenspace will be all vectors $v \in \mathbb{R}^n$ satisfying $Av = \lambda v$.

3. Diagonalization

Why do we like eigenvalues and eigenvectors? If $\{v_1, v_2, ..., v_n\}$ is a basis of f contained only of eigenvectors (i.e. for any $0 \le i \le n$: $f(v_i) = \lambda_i v_i$) then the corresponding matrix of f with respect to this basis is just

$$\begin{pmatrix} \lambda_1 & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & . & 0 \\ 0 & 0 & \lambda_3 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

i.e. a matrix whose only non zero entries sit in its main diagonal and these are exactly the eigenvalues. Finding such a basis s called diagonalizing f. A linear map that has such a basis is called diagonalizable.

3.1. Corollary: if f is diagonalizable then $Det(f)$ is the product of its eigenvalues.

3.2. Example: let us diagonalize f given in some basis $\{x, y\}$ of $\mathbb{R}^2$ by $f(x) = x - y, f(y) = y - x$. The corresponding matrix A is $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then $P_f(x) = P_A(x) = Det(xId_{n\times n} - A) = Det\begin{pmatrix} x-1 & 1 \\ 1 & x-1 \end{pmatrix} = (x-1)^2 - 1 = x(x-2)$ and we are interested in its roots so $\lambda_1 = 0$, $\lambda_2 = 2$. Now let us try to find a basis consisting of eigenvectors only: $Av_1 = 0v_1$ so we can solve this and find that $x+y$ is an eigenvector of the eigenvalue 0. Similarly we can take $x - y$ that is an eigenvector of the eigenvalue 2. We conclude that with respect to the basis $\{x+y, x-y\}$ f corresponds to the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

3.3. Exercise: let $\{x, y\}$ be some basis of $\mathbb{R}^2$. Define a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = x + 3y$, $f(y) = 5x + 7y$. Diagonalize f (in fact you know the answer from exercise 3.6 of homework 6, please show the explicit calculation).

3.4. Exercise: Diagonalize the matrix $\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{3}{2} & -1 \\ 0 & 0 & 3 \end{pmatrix}$ (i.e. this matrix corresponds to some linear map f with respect to some basis - diagonalize f).

Warning! Not all linear maps are diagonalizable. The following exercise shows it:

3.5. Exercise: let $\{x, y\}$ be some basis of $\mathbb{R}^2$. Define a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = x$, $f(y) = x + y$. Show that f has only one eigenvalue (and find it) and show that the eigenspace corresponds to this eigenvalue is one dimensional. Conclude that f is not diagonalizable.

In fact, some linear maps have no eigenvectors at all. The following exercise shows it:

3.6. Exercise: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be rotation around the origin in some angle $\alpha \not\equiv 0 \ (mod\ 2\pi)$ (i.e. f is not the identity function). Prove that f has no (real) eigenvalues, i.e. that there is no $\lambda \in \mathbb{R}$ and a non-zero $v \in \mathbb{R}^n$ such that $f(v) = \lambda v$.

4. Fixed points

4.1. Definition: an eigenvector corresponding to the eigenvalue 1 is called a fixed point of f (these are exactly all the vectors that f "does nothing to").

Fixed points of a linear map will be of great importance to us.

4.2. Claim: fix some basis of $\mathbb{R}^n$ (and now you have the notion of length). If λ is an eigenvalue of f then f has an eigenvector v corresponding to λ such that the length of v is 1.

Proof: let $v \in \mathbb{R}^n$ be non-zero vector satisfying $f(v) = \lambda v$ (such v exists as λ is an eigenvalue of f). Define $v' := \frac{v}{|v|}$, then $f(v') = f(\frac{v}{|v|}) = \frac{\lambda v}{|v|} = \lambda \frac{v}{|v|} = \lambda v'$, and $|v'| = |\frac{v}{|v|}| = \sqrt{< \frac{v}{|v|}, \frac{v}{|v|} >} = \sqrt{\frac{<v,v>}{|v|^2}} = \sqrt{\frac{|v|^2}{|v|^2}} = 1$. Thus we showed that v' satisfies the conditions of the claim. $\square$

4.3. Corollary: fix some basis of $\mathbb{R}^n$ (and now you have the notion of length). If f has a fixed point (i.e. 1 is an eigenvalue of f) then f has a fixed point of length 1.

4.4. Exercise: $\{e_1, e_2, e_3\}$ be some basis of $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given in this basis by the matrix $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. Find a fixed point of length 1 of f (i.e. find a vector $v \in \mathbb{R}^3$ such that $f(v) = v$ and $|v| = 1$).

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