ENGINEERING MATHEMATICS - TUTORIAL AND HOMEWORK 9

Please submit all exercises below in hard copy (either in English or in Hebrew) next Thursday, January 7^{th}, in the tutorial. Sections marked as bonus are not mandatory.

1. Norms on $\mathbb{R}^n$

For this chapter we fix a basis $\{e_1, e_2, ..., e_n\}$ of $\mathbb{R}^n$. Thus for any vector $v \in \mathbb{R}^n$ we have coordinates $v_1, v_2, ..., v_n$.

We saw that using the inner product we can define the norm of a vector $v \in \mathbb{R}^n$ by $|v| = \sqrt{\Sigma_{i=1}^n v_i^2} = (\Sigma_{i=1}^n v_i^2)^{\frac{1}{2}}$. Let us denote this norm function by $|\cdot|_2$, i.e. we add a lower index 2, and from now on we write $|v|_2 = \sqrt{\Sigma_{i=1}^n v_i^2} = (\Sigma_{i=1}^n v_i^2)^{\frac{1}{2}}$. This norm is often called the Euclidean norm or L^2 norm, and this is how we are used to "measure" length in $\mathbb{R}^n$. However, there are many ways to do so, i.e. many functions from $\mathbb{R}^n$ to $\mathbb{R}$ that will satisfy the "natural" conditions we should demand. Let us define these conditions:

1.1. Definition: a norm on $\mathbb{R}^n$ is a function $N : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions for any $\lambda \in \mathbb{R}$ and any $v, w \in \mathbb{R}^n$:

(1) $N(\lambda v) = |\lambda| N(v)$ (homogeneity).
(2) $N(v + w) \leq N(v) + N(w)$ (triangle inequality).
(3) if $N(v) = 0$ then $v = 0$.

1.2. Claim: if N is a norm then for any $v \in \mathbb{R}^n$: $N(v) \geq 0$.

Proof: by condition (1) we have $N(0) = 0$ and for any $v \in \mathbb{R}^n$: $N(v) = N(-v)$. By condition (2) we have $0 = N(0) = N(v+(-v)) \leq N(v)+N(-v) = N(v)+N(v) = 2N(v)$, thus $N(v) \geq 0$. □

1.3. Examples:

(1) $|\cdot|_2$ is a norm.
(2) let us define the L^1 norm by $|v|_1 = \Sigma_{i=1}^n |v_i|$. This is sometimes called "the Manhattan taxi drivers norm" (think why).
(3) let $1 \leq p \in \mathbb{R}$ and define the L^p norm by $|v|_p = (\Sigma_{i=1}^n |v_i|^p)^{\frac{1}{p}}$. Note that for $p = 2$ we get example (1) and when $p = 1$ we get example (2).
(4) let us define the L^∞ norm by $|v|_\infty = max\{|v_i|\}_{i=1}^n$. This is sometimes called the maximum norm.

1.4. Definition: let N be a norm on $\mathbb{R}^n$. The set $B_N := \{v \in \mathbb{R}^n | N(v) < 1\}$ (respectively $\{v \in \mathbb{R}^n | N(v) \leq 1\}$) is called the open (resp. closed) unit ball corresponding to N.

1.5. Definition: We call a set A in $\mathbb{R}^n$ convex if for any two vectors $v_1, v_2 \in A$ and any real number $0 \leq \lambda \leq 1$: $\lambda v_1 + (1 - \lambda) v_2 \in A$..

1.6. Exercise: let N be a norm on $\mathbb{R}^n$. Prove that B_N is convex.

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1.7. Exercise: take your favorite basis $\{x, y\}$ of $\mathbb{R}^2$ and draw the "unit spheres" of the norms L^2, L^1 and L^∞, i.e. plot the set of all points $v \in \mathbb{R}^2$ satisfying $|v|_2 = 1$, the set of all points $v \in \mathbb{R}^2$ satisfying $|v|_1 = 1$ and the set of all points $v \in \mathbb{R}^2$ satisfying $|v|_\infty = 1$. Explain your answers.

1.8. Exercise: let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that 1 is an eigenvalue of f, and let $N : \mathbb{R}^n \to \mathbb{R}$ be a norm on $\mathbb{R}^n$. Prove that f has a fixed point $v \in \mathbb{R}^n$ satisfying $N(v) = 1$ (hint: this is very similar to claim 4.2 of homework number 8).

Remark: the subject of norms is much richer than what we discussed, however the examples above will be sufficient for our needs. We will especially use the L^1 norm.

2. Basic probability (reminder)

2.1. Exercise: you roll a standard fair dice with 6 faces (such that the probability of each number to occur is equal) and choose a vacation destination using the following process: first roll the dice and choose the country to visit by:

$$1 - England, 2 - France, 3 - Italy, 4 - Thailnad, 5 - Canada, 6 - Finland.$$

Afterwards roll the dice again, denote the outcome of this roll by N, and choose the city to visit by the following cases:

(1) if you are in England visit London if $N \equiv 0(mod\ 3)$ and visit Manchester otherwise.
(2) if you are in France visit Chamonix if $N \equiv 0(mod\ 2)$ and visit Paris otherwise.
(3) if you are in Italy go to Rome.
(4) if you are in Thailand go to Chiang-Mai.
(5) if you are in Canada go to Quebec city.
(6) if you are in Finland roll the dice again (forget N) and denote the outcome of this new roll by M. If $M > 1$ go to Helsinki, otherwise go to Rovaniemi.

What is the probability you will find yourself having a vacation in a European capital?

2.2. Example ("the birthday paradox"): Assume N people are in a room, each have a birthday date uniformly distributed in the year (i.e. the probability of a person having a birthday in a given day is independent of the day, thus it is just $\frac{1}{365}$ - we forget February 29^{th} for this exercise), and the birthday dates are all independent. What is the probability that at least two people in the room share their birthday date?

Solution: instead of calculating the probability that at least two people in the room share their birthday date (denote this number by X), we will calculate the probability that no two people in the room share their birthday date (denote this number by Y). It is clear that $X = 1 - Y$.

Let us line all N people in a row. The first person can have any birthday date. The second will now have 364 options (in order not to have the same date as the first), so with probability $\frac{364}{365}$ the event "no two people in the room share their birthday date" may still happen. The third will now have 363 options (in order not to have the same date as the first or the second), so with probability $\frac{363}{365}$ the event "no two people in the room share their birthday date" may still happen. The fourth will now have 362 options (in order not to have the same date as the first or the second or the third), so with probability $\frac{362}{365}$ the event "no two people in the room share their birthday date" may still happen. So on we proceed untill the N^{th} person will have 365-$(N-1)$ options, and eventually with probability $\frac{365-(N+1)}{365}$ the event "no two people in the room share their birthday date" may still happen. As we need all events above to happen, multiplying the above probabilities we get exactly Y:

$$Y = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(N-1)}{365} = \Pi_{i=1}^{N-1} \frac{365-i}{365}.$$

Eventually we get that the probability that at least two people in the room share their birthday date is

$$X = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(N-1)}{365} = 1 - \Pi_{i=1}^{N-1} \frac{365-i}{365}.$$

If we take $N = 23$ we get that X is approximately 0.507, i.e. there is a probability of more than 50% to find two people in the room that share their birthday date. If we take $N = 60$ we get a probability of more than 99%. These results are sometimes called "the birthday paradox".

Remark: This solution is not a rigourous mathematical proof, however it is not difficult to construct one using it (the calculations are all correct but should be explained more carefully - if you read the English Wikipedia article about "the birthday paradox" beware of a small mistake in these explanations).

2.3. Exercise: you construct a six digit number in the following way: you roll a standard fair dice with 6 faces six times. The first digit is result of the first roll, the second is the second result and so on.

(1) What is the probability to get the number 123456?
(2) What is the probability to get a number that all of its digits are pairwise different (i.e. a number that each digit 1,2,3,4,5,6 appears in it exactly once)?
(3) What is the probability to get an even number?
(4) What is the probability to get a number greater than 311111?

2.4. Recall that for a random variable X that takes the value X_i with probability P_i we define its expected value by $E(X) := \Sigma_i(p_i \cdot x_i)$ (if this number is finite). If X only takes finitely many $x_i's$ with probabilities different than zero it is always defined.

2.5. Exercise: what is the expected value of a standard fair dice with 6 faces.

2.6. Exercise: you roll two standard fair dices with 6 faces each. If both numbers that came up are the same we say you got a double. Now you repeat this process 3 times. What is the probability to get 3 doubles? Playing Monopoly this will send you to jail.

2.7. Exercise: we take a coin that falls on "heads" with probability p and on "tails" with probability $1 - p$.

(1) We play the following game: you flip the coin and win w shekels if it fell on "heads" and loss l shekels if it fell on "tails". What is the expected value of the game (thinking of the game as a random variable taking w with probability p and taking $-l$ with probability $1 - p$)? This can be interpreted as the average number of shekels you win each time you play the game.

(2) We play the following game: you flip the coin over and over, up to four times, until you get "heads". The first time you get "heads" the game is over. If you got "heads" in the first chance you get w_1 shekels, if you got "heads" in the second chance you get w_2 shekels, if you got "heads" in the third chance you get w_3 shekels, and if you got "heads" in the fourth chance you get w_4 shekels. If all four times you got "tails" you win nothing. What is the expected value of the game? There is no need to simplify the expression you got.

(3) Take $p = \frac{1}{2}$. We play the following game: you flip the coin over and over until you get "heads". The first time you get "heads" the game is over. If the first time was after n tries you get 2^n shekels. What is the expected value of the game?

2.8. Intuition test: there are 13 students in the Rothschild-Weizmann program. In order to choose a student who will stay after class to clean the classroom we do the following process: we take 13 paper notes and write "you lost" on one of them only. We fold the notes and put them in a bowl. Each student takes a note out of the bowl and keeps it. The one who got the "you lost" note stays after class. In order to have the best probability of not staying after class - would you prefer to be the first one who takes a note? Or maybe the last? After giving some thought to this you can find the answer in this (Hebrew!) article. This is a nice implication of probability theory in everyday life.

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