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## INTRO TO SINGULARITIES

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### EXERCISE 1

**1. Resultants.** For two homogeneous polynomials  $f, g \in \mathbb{C}[x, y]$ ,  $\deg f = m$  and  $\deg g = n$ , define a linear map  $L_{f,g} : \mathbb{C}_{n-1}[x, y] \times \mathbb{C}_{m-1}[x, y] \rightarrow \mathbb{C}_{m+n-1}[x, y]$  by  $L_{f,g}(a, b) = af + bg$  (where  $\mathbb{C}_k[x, y]$  denotes the space of homogeneous polynomials in  $x, y$  of degree  $k$ ).

- (1) Show that the two linear spaces have the same dimension. Choosing the standard basis  $\{x^i y^{k-i}\}_{i=0}^k$  in  $\mathbb{C}_k[x, y]$ , one can define resultant as  $R(f, g) = \det L_{f,g}$  (invariant object is the map itself, but  $R(f, g)$  is a classical notion). Write the matrix of  $L_{f,g}$  in this basis.
- (2) Prove that  $R(f, g)$  is a homogeneous polynomial in coefficients of  $f, g$ . What is its degree?
- (3) Prove that  $R(f, g)$  is a homogeneous polynomial in  $(\{\lambda_i\}, \{\mu_j\})$ , where  $f = \prod_i (x - \lambda_i y)$  and  $g = \prod_j (x - \mu_j y)$  (here we have to assume that both  $f$  and  $g$  are not divisible by  $y$ ). What is its degree? (*Hint*: Use definition of homogeneity).
- (4) Prove that  $R(f, g) = 0$  if and only if  $f$  and  $g$  have common linear factor (look on the image and kernel of  $L_{f,g}$  correspondingly).
- (5) Assume that a polynomial  $p \in \mathbb{C}[x_1, \dots, x_n]$  vanishes on  $\{x_1 = 0\}$ . Prove that  $p = x_1 \tilde{p}$  for some another polynomial  $\tilde{p}$  (a variant of Hadamard lemma). Conclude that  $R(f, g) = c \prod (\lambda_i - \mu_j)$ .
- (6) Prove that  $R(f, gh) = R(f, g)R(f, h)$ .
- (7) Prove that  $R(f + \phi g, g) = R(f, g)$  for a homogeneous polynomial  $\phi$ ,  $\deg \phi = m - n$ .

#### 0.1. 2. Multiplicity.

- (1) Compute dimension of  $\mathbb{C}[[x, y]]/\langle y^2 - xy, xy - x^3 \rangle$ .
- (2) On the lecture the multiplicity of zero of polynomial  $f$  at the origin was defined as "the number of zeros of a small perturbation of  $f$  near the origin". Write explicitly the formal definition.
- (3) Multiplicity of a common isolated zero of two polynomials  $f, g$  situated at the origin is defined as  $\mathbb{C}[[x, y]]/\langle f, g \rangle$ , and can be also defined as the number of points in  $\{f = \varepsilon_1, g = \varepsilon_2\}$  close to the origin for small generic  $\varepsilon_1, \varepsilon_2$ . How this is correlated with the receipt of counting multiplicity of curves intersection through "good parameterization"? Give an example showing that considering  $(\varepsilon_1 = 0, \varepsilon_2 = \text{any})$  is not sufficient for *arbitrary* polynomials  $f, g$ .

#### 0.2. 3. Polar curve.

- (1) Show that for the quadric  $q = X^2 + Y^2 - Z^2$  the duality in the affine chart  $\mathbb{C}^2 \cong \{Z \neq 0\}$  is  $(x_0, y_0) \leftrightarrow \{xx_0 + yy_0 - 1 = 0\}$  (where  $x = X/Z, y = Y/Z$ ).
- (2) Compute polar curve of the curve  $(y - 1)^2 = x^3$ .