MANIFOLDS: FALL 2014
EXERCISE 10

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Problem 1. *Choose 3 from 2.4.1, 2.4.2, 2.4.18, 2.4.13, 2.4.14, 2.4.15.*

Problem 2. *As an application of 2.4.15, prove Borsuk-Ulam theorem in dimension one: assume that* $f : \mathbb{S}^1 \to \mathbb{S}^1$ *is odd, i.e.* $f(x + \pi) = f(x) + \pi, x \in \mathbb{R}/2\pi\mathbb{Z}$ *(or, equivalently,* $f(-x) = -f(x)$ *if we consider* $\mathbb{S}^1$ *as a subset of* $\mathbb{R}^2$*). Then* $\deg_2 f = 1$*. Here is a plan: assume* $f(0) \neq 0$*. Let* $M = \mathbb{S}^1 \times \mathbb{S}^1$*.*

1. *Construct* $Z \subset M$ *a smooth submanifold coinciding with* $\mathbb{S}^1 \times \{0\} \cup \{0\} \times \mathbb{S}^1$ *outside a small neighborhood of* $(0, 0)$ *(replace a cross with two "hyperbolas").*
2. *Prove that* $\Delta = \{(x, x)|x \in \mathbb{S}^1\} \subset M$ *is cobordant to* Z*.*
3. *Let* Γ *be a graph of* f*. Show that* $I_2(\Gamma, \Delta) = 0$*.*
4. *Conclude that* $I_2(\Gamma, Z) = 0$*.*
5. *But* $I_2(\Gamma, Z) = I_2(\Gamma, \mathbb{S}^1 \times \{0\}) + I_2(\Gamma, \{0\} \times \mathbb{S}^1)$*,*
6. $I_2(\Gamma, \{0\} \times \mathbb{S}^1) = 1$*,*
7. *and* $I_2(\Gamma, \mathbb{S}^1 \times \{0\}) = \deg_2 f$*.*

Remark: this proof generalizes to higher dimensions.

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