MANIFOLDS: FALL 2014
EXERCISE 11

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ABSTRACT. 2.5-2.6 of Guillemin-Pollack.

Problem 1. *Prove that the eight-figure* $\gamma_0(t) = (\cos t, \sin 2t)$ *is not homotopic to a circle* $\gamma_1 = (\cos t, \sin t)$ *in the class of immersed curves (i.e. such that each* γ_t *is an immersion of* $\mathbb{S}^1$ *for* $t \in [0,1]$*). Hint: follow* $W_2(\gamma_t', 0)$.

Problem 2. *Construct a continuous mapping* $f : T^2 \to \mathbb{R}^2$ *such that* $f(x)$ *is never equal to* $f(-x)$.

Application of the Borsuk-Ulam theorem (see the link to the excellent book of Matousek) to combinatorics are numerous.

Problem 3. *Lyusternik-Shnirelman theorem*

(1) *Whenever* $\mathbb{S}^n$ *is covered by* $n+1$ *sets* A_1, A_2,...,A_{n+1}, *each* A_i *closed, there is an* i *such that* $A_i \cap (-A_i) \neq \emptyset$ *(Hint:* $\mathrm{dist}(\cdot, A_i), i = 1, ..., n$ *are continuous functions on* $\mathbb{S}^n$*).*
(2) *Give an example showing that* $n+1$ *cannot be replaced by* $n+2$.

Problem 4. *Ham sandwich theorem: Let* $\mu_i, i = 1, ..., d$ *be finite absolutely continuous Borel measures on* $\mathbb{R}^d$. *Then there exists an affine hyperplane* $h \subset \mathbb{R}^d$ *such that* $\mu_i(h_+) = \frac{1}{2}\mu_i(\mathbb{R}^d)$ *for* $i = 1, 2, ..., d$, *where* h_+ *denotes one of the half-spaces bounded by* h.

Problem 5. *Prove that any measurable set in the plane can be dissected into four equal parts by two lines.*

Remark: this cannot be generalized to $\mathbb{R}^5$: any 5 hyperplanes cut the so-called moment curve $(t, t^2, ..., t^5)$ into at most 25 pieces, which is less than $2^5 = 32$ (why this is a counterexample?).

By taking a limit, one obtain a Ham sandwich theorem for a d-tuple of finite sets of points: if $A_i, i = 1, ..., d$ are finite subsets of $\mathbb{R}^d$ such that no $d+2$ points of $\cup A_i$ are collinear then there exist a hyperplane bisecting all of them simultaneously. As an application, there is a proof of Necklace theorem on p.54-55 of
`http://www.maths.ed.ac.uk/ aar/papers/matousek1.pdf`

Date: January 15, 2015.