MANIFOLDS: FALL 2014
EXERCISE 8

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Abstract. 2.3 by Guillemin, Pollack, 2.2-beginning of 2.3 by Golubitsky, Guillemin

Problem 1. *Let Y be some two-dimensional non-compact manifold, $Z \subset Y$ be its zero-dimensional submanifold and let $f : \mathbb{R} \to Y$ be a smooth mapping. Construct an arbitrarily small deformation f_ϵ of f such that its image does not intersect Z.*

Problem 2. *Let Y be a non-compact manifold. For which $f \in C^\infty(Y, \mathbb{R})$ the mapping $\mathbb{R} \to C^\infty(Y, \mathbb{R})$, $\lambda \to \lambda f$, is continuous?*

Problem 3.
(1) *Construct some scalar product on the fibers $J^k_{(x,y)}(X, Y)$ of the natural projection $\pi : J^k(X, Y) \to X \times Y$ smoothly depending on (x, y) (Hint: Use charts to construct them locally, glue them together by partition of unity, show that the result is still a scalar product in each fiber).*
(2) *Show that for compact X, Y the norms $\| \|_k$ induced by such scalar products are all equivalent: $\forall \| \|_k, \| \|'_k \ \exists c, C > 0 : c\| \|_k \le \| \|'_k \le C \| \|_k$.*

Define $B_\delta(f) = \{g \in C^\infty(X, Y | \forall x \in X\, d(j^k_x f, j^k_x g) < \delta(x)\}$, where $d(\cdot, \cdot)$ is some metric on $J^k(X, Y)$ compatible with its topology (it exists by some general results) and $\delta(x)$ is a positive continuous function on X. These sets give another basis of the Whitney topology in $C^\infty(X, Y)$ (see Golubitsky, Guillemin).

Problem 4. *Prove that $C^\infty(X, Y)$*
(1) *is a first-countable space for compact X.*
(2) *and is not first-countable space for non-compact X.*

Problem 5. *Show that $C^\infty(X, \mathbb{S}^1)$ is metrizable for a compact X: check that the metric $\|f - g\|_\infty = \sum 2^{-k} \arctan \|f - g\|_k$ produces the same topology. Here $\|f - g\|_k = \max_{(x, f(x)-g(x)) \in X \times \mathbb{S}^1} \left[|f(x) - g(x)| + \|j^k_x (f - g)\| \right]$, where $\| \|_k$ is some euclidean norm in $J^k_{(x,f(x))}(X, \mathbb{S}^1)$ defined in Problem 3.*

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