MANIFOLDS: SPRING 2015
EXERCISE 1

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Problem 1. *In Warner's book submanifolds are defined differently: a submanifold of N is a pair (M, ϕ), where M is a manifold and $\phi : M \to N$ is an immersion, and not an embedding. Let us call such pairs "immersed submanifold".*

1. *Assume $\gamma : (a, b) \to N$ is a curve such that $\gamma\left((a, b)\right) \subset \phi(M)$ (where (M, ϕ) is an immersed submanifold). Show that not necessarily $\dot\gamma(t) \in T_{\gamma(t)}\phi(M)$ (Hint: first, define $T_{\gamma(t)}\phi(M)$).*
2. *Show that an integral curve of a vector field is always an immersed submanifold, but could be not a submanifold,*

Problem 2. *Show that any smooth vector field on a compact manifold is complete.*

Problem 3. *Let $\phi : M \to N$ be a smooth mapping, and let X be a smooth vector field on on M. A smooth vector field Y on N is called "related to X if $d\phi(X(p))) = Y(\phi(p))$ for all $p \in M$.*

1. *Assume $d\phi(X(p)) = d\phi(X(q))$ whenever $\phi(p) = \phi(q)$. Is there a smooth vector field Y on N related to X?*
2. *Prove that if $X_{1,2}$ are two smooth vector fields on M, and the vector fields $Y_{1,2}$ are related to $X_{1,2}$ correspondingly, then $[X_1, X_2]$ is related to $[Y_1, Y_2]$*

Problem 4. *Let v_i be smooth vector fields on M_i, $i = 1, 2$. They define two vector fields $\tilde v_1 = (v_1, 0)$ and $\tilde v_2 = (0, v_2)$ on $M_1 \times M_2$. Show that $[\tilde v_1, \tilde v_2] = 0$.*

Problem 5. *Let $v_{1,2}$ be two linear vector fields on $\mathbb{R}^d$, i.e. $v_i = A_i x$ for some matrices $A_i \in \mathcal{M}_{n\times n}(\mathbb{R})$, $i = 1, 2$.*

1. *Compute the flow X_s of v_1. Show that v_1 is complete.*
2. *Compute $[v_1, v_2]$.*
3. *Give an example of $A_{1,2}$ such that the minimal involutive distribution D such that $v_i \in D$ has dimension d.*

Problem 6. *Assume v is a smooth vector field on $\mathbb{R}^d$, and $\|v(x)\| \le K\|x\|$. Show that v is complete.*

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