MANIFOLDS: SPRING 2015
EXERCISE 2

DMITRY NOVIKOV

Problem 1. *A Godbillon-Vey sequence for a one-form ω_0 is a sequence of one-forms $\omega_1, ..., \omega_n, ...$ such that*

$$d\omega_n = \omega_0 \wedge \omega_{n+1} + \sum_{i=1}^{n+1} \binom{n}{k} \omega_i \wedge \omega_{n+1-i}. \tag{1}$$

(1) *If ω_0 has a Godbillon-Vey sequence then it is integrable, i.e. $\omega_0 \wedge d\omega_0 = 0$,*
(2) *If ω_0 is integrable, then for any vector field X such that $\omega_0(X) \equiv 1$ the sequence $\omega_{i+1} = L_X \omega_i$ is a Godbillon-Vey sequence for ω_0,*
(3) *Assume $\omega_1, ..., \omega_n, ...$ is a Godbillon-Vey sequence for a one-form ω_0. Write a Godbillon-Vey sequence for a one-form $f\omega_0$, where f is some smooth function.*
(4) *Assume that G is a transcendental over K solution of the PDE*

$$dG + \frac{G^n}{n!}\omega_n + ... + \omega_0 = 0, \tag{2}$$

where all ω_i have coefficients in some field K. Show that $\omega_1, ..., \omega_n, 0, 0, ...$ is a Godbillon-Vey sequence for the ω_0.
(5) *Let $\omega_0 = xdy + \pi y dx$. Find a Godbillon-Vey sequence of forms with rational coefficients for ω_0 of length 2 (i.e. $0 = \omega_2 = ...$).*

Further reading: An extension of a differentiable field K is called Liouvillian if it is obtained by a finite sequence of extensions $K \subset K(G)$ with G either algebraic or a transcendent solution of $dG = G\gamma_1 + \gamma_0$, with γ_i being one-forms with coefficients in K.

A nice paper "Suites de GodbillonVey et intégrales premières" by Guy Casale, http://www.sciencedirect.com/science/article/pii/S1631073X02026195# gives, among other, a short proof of the Singer theorem: ω_0 with coefficients in K has a Liouvillian first integral H (i.e. $dH = M\omega_0$) if and only if it admits a Godbillon-Vey sequence for ω_0 of length 2 of forms with coefficients in K (and then dM has coefficients in K). This is a typical Galois theory type result: if an equation has a solution after some sequence of base field extension, then it has it after just several extension (in this case two).

Problem 2. (1) *A form ω is called* decomposable *if $\omega = \omega_1 \wedge ... \wedge \omega_k$ for some one-forms ω_i. IS it true that any k-form is decomposable?*
(2) *Is it true that $\omega \wedge \omega = 0$ for any form ω?*

Problem 3. *Let L be a d-dimensional linear space with a non-degenerate scalar product $(\, , \,)$ and orientation.*

Date: April 27, 2015.

(1) *Construct a scalar product on* $\Lambda_* L$*. In particular, there is an isomorphism* $*_k : \Lambda_k L \to \Lambda_{d-k} L$.

(2) *Compute* $*_k *_{d-k}$*. Prove that* $(v, w) = *(w \wedge *v) = *(v \wedge *w)$.

We denote by $*^k : \Lambda^k L \to \Lambda^{d-k} L$ *the isomorphism dual to* $*_{d-k}$ *and by* $\beta_k : \Lambda^k L \to \Lambda_k L$ *another isomorphism (which one?)*

Problem 4. *Here we consider the forms in* $\mathbb{R}^3$ *with non-constant coefficients (i.e.* $E^*(\mathbb{R}^3)$*).*

(1) *Prove that any 2-form* ω *in* $\mathbb{R}^3$ *can be represented as* $i_v \Omega$*, where* $\Omega = dx \wedge dy \wedge dz$ *is the volume form. Denote this isomorphism as* $i_\Omega : \mathfrak{X}(\mathbb{R}^3) \to E^1(\mathbb{R}^3)$*. What is the relation between* $i_\Omega^{-1}(\omega)$ *and* $\beta_2 \omega$*?*

(2) *Write in coordinates* $i_\Omega^{-1}(d(\beta_1^{-1}(v)))$*. And* $\frac{d(i_\Omega^{-1}(v))}{\Omega}$.

(3) *What is* $i_\Omega^{-1}(\beta_1(u) \wedge \beta_1(v))$.

Problem 5. *Let* $M \subset L$ *be a linear subspace of dimension* k, $\dim L = d$*. Then* $\Lambda_k M \subset \Lambda_k L$ *is a line and therefore we have the Plcker embedding* $\psi : Gr(k, L) \to P(\Lambda_k L)$*. Show that it is a smooth immersion, but not submersion. Show that it is embedding.*