MANIFOLDS: SPRING 2015
EXERCISE 3

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Problem 1 (Hamiltonian systems)**.** *Let M be a smooth $2n$-dimensional manifold. A 2-form $\omega \in E^2(M)$ is called a symplectic form on M if it is closed (i.e. $d\omega = 0$) and non-degenerate (i.e. for any $x \in M$ and any $\xi \in T_xM$ there exists $\eta \in T_xM$ such that $\omega(\xi, \eta) \neq 0$).*

(1) *Let $H \in C^\infty M$ be a smooth function. Define a vector field v_H on M by $\omega(v_H, \xi) = dH(\xi)$ for all $\xi \in TM$ (this is the* Hamiltonian vector field *corresponding to H). Show that $L_{v_H}\omega = 0$. Deduce that $L_{v_H}\omega^{\wedge^n} = 0$, i.e. that the Hamiltonian flows preserve the volume.*
(2) *Define Poisson bracket $\{H_1, H_2\}$ as $\{H_1, H_2\} = dH_1(v_{H_2})$. Show that $\{H_1, H_2\} = -\{H_2, H_1\}$ and Jacobi identity for $\{H_1, H_2\}$.*
(3) *A Hamiltonian vector field v_{H_1} is called* integrable *if there exist functions $H_2, ...H_n$ such that $\{H_i, H_j\} \equiv 0$ for $i, j = 1, ..., n$ and v_{H_i} are linearly independent. Show that H_i are constant on trajectories of v_{H_1} (i.e. that $L_{v_{H_j}} H_i = 0$ for $i, j = 1, ..., n$).*
(4) *Assume that v_{H_i} are linearly independent at each point of some common level set $\cap\{H_i = c_i\}$. Let $p \in \Sigma$, where Σ is a connected component of this common level set, and define $F_p(t_1, ..., t_n) = X^1_{t_1} \circ ... \circ X^n_{t_n}(p)$, where X^i_t is the flow of v_{H_i}. Show that*

$$F(F(p, t_1, ..., t_n), t'_1, ..., t'_n) = F(p, t_1 + t'_1, ..., t_n + t'_n).$$

Show that F is a local diffeomorphism, and therefore is a covering $\mathbb{R}^n \to \Sigma = F(\mathbb{R}^n)$, and $F^{-1}(p)$ is a (shift of a) discrete subgroup of $\mathbb{R}^n$. Deduce that $\Sigma = \mathbb{S}^1 \times ... \times \mathbb{S}^1 \times \mathbb{R} \times ... \times \mathbb{R}$.

Problem 2 (Riemann manifolds)**.** *Let M be a smooth n-dimensional manifold. M is called a* Riemannian *manifold if it is equipped with a non-degenerate scalar product $\langle\, ,\, \rangle_x$ on each T_xM, smoothly dependent on $x \in M$ (the so-called* Riemannian structure*).*

(1) *Show that every smooth manifold admits* some *Riemannian structure (One way to prove it is as follows: choose some covering by charts, choose some scalar products in each chart and glue them together using partition of unity. The only non-trivial question is why the result is positive-definite).*
(2) *Any submanifold of a Riemannian manifold is a Riemannian manifold itself (in particular, this implies the previous claim).*
(3) *An orientable Riemannian manifold has a canonical volume form: define $\omega \in E^n(M)$ by $\omega(e_1, ..., e_n) = 1$ for some positively oriented orthonormal (w.r.t. $\langle\, ,\, \rangle$) basis of T_xM. Check that this is a well-defined form.*

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Problem 3 (* operator)**.** *Let M be an oriented Riemannian manifold.*

(1) *A scalar product $\langle\, , \,\rangle_x$ on the linear space T_xM defines the so-called* musical isomorphisms $\flat : T_xM \to T_x^*M$ *and* $\sharp = \flat^{-1} : T_x^*M \to T_xM$*, and therefore an identification* $\flat^{\wedge p} : \Lambda^p(T_x^*M) \to \Lambda^p(T_x^*M)$*. Prove that it defines a scalar product on* $\Lambda^p(T_x^*M)$*, with* $\{\eta_{i_1} \wedge ... \wedge \eta_{i_p}\}$ *as an orthonormal basis (where* $\{\eta_1, ..., \eta_n\}$ *is an orthonormal basis of* T_x^*M*).*

(2) *Let* $\{\eta_1, ..., \eta_n\}$ *be a positively oriented orthonormal basis of* T_x^*M*. Define* $* : \Lambda(T_x^*M) \to \Lambda(T_x^*M)$ *as*

$$*(1) = \eta_1 \wedge ... \wedge \eta_n, \quad *(\eta_1 \wedge ... \wedge \eta_n) = 1 \tag{1}$$

$$*(\eta_{i_1} \wedge ... \wedge \eta_{i_k}) = (-1)^\sigma \eta_{i_{k+1}} \wedge ... \wedge \eta_{i_n}, \tag{2}$$

where $\sigma = 1$ *if the transposition* $i_1...i_n$ *is odd and* $\sigma = 0$ *if it is even. Show that this definition is independent on the choice of* η_i *and that* $** = (-1)^{p(n-p)}$ *on* $\Lambda^p(T_x^*M)$*.*

(3) *Alternatively,* $\Lambda^p(T_x^*M)$ *is dual to* $\Lambda^{n-p}(T_x^*M)$*: for* $\omega \in \Lambda^p(T_x^*M)$*,* $\eta \in \Lambda^{n-p}(T_x^*M)$ *define* $\langle \omega, \eta \rangle = \omega \wedge \eta(v_1, ..., v_n)$*, where* $\{v_i\}$ *is some positively oriented orthonormal basis of* T_xM*. Therefore* $\Lambda^p(T_x^*M)$ *is isomorphic to* $\Lambda^{n-p}(T_x^*M)$*, as both are dual to* $\Lambda^p(T_x^*M)$*. Check that this isomorphism is the* $*$ *operator defined above.*

(4) *Compute* $*d*d(f)$ *for* $f \in \mathcal{C}^\infty(\mathbb{R}^n)$*.*

Problem 4. *Chapter 6.1-6.8 of Warner (without the proof of Hodge theorem).*