FINAL EXAM FOR THE COURSE MATHEMATICS IN THE DAILY LIFE

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Abstract. This is take home exam. Due date is February 29, 2012. Good luck and may the Force be with you!

Problem 1. Show that a transformation T of the plane or of the space preserving distances and angles and having a fixed point is a linear transformation. Suggestion:

(a) Start by proving that the transformation preserves the sum of two vectors, using that the sum $v_1 + v_2$ of the two vectors is constructed as the diagonal of the parallelogram with sides v_1 and v_2.

(b) Show now that for any vector v and any $c \in \mathbb{R}$, then $T(cv) = cT(v)$. Make the argument in several steps:
- Prove the assertion for $c \in \mathbb{N}$.
- Prove the assertion for $c \in \mathbb{Q}$.
- Show that T is uniformly continuous. Use this to prove it for $c \in \mathbb{R}$. Indeed, if $c = \lim_{n\to\infty} c_n$ with $c_n \in \mathbb{Q}$, and if T is continuous, then $T(cv) = \lim_{n\to\infty} T(c_n v)$.

Problem 2. Consider a rotation by the angle $+\pi/4$ about the axis v_1 determined by $v_1 = (1/3, 2/3, 2/3)^t$. Using the basis $\mathcal{B} = \{v_1, v_2, v_3\}$, where $v_1 = (1/3, 2/3, 2/3)^t$, $v_2 = (2/3, -2/3, 1/3)^t$, and $v_3 = (2/3, 1/3, -2/3)^t$, give the matrix describing this rotation expressed in the standard basis.

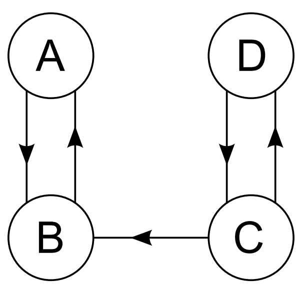

Figure 1. Problem 3

Problem 3. Consider the web illustrated above.

(a) Which of the pairs of pages, (A, B) or (C, D), will be given a greater rank by the simplified PageRank algorithm?

(b) Find the page ranking assigned by the simplified PageRank algorithm.

(c) Find the stationary regime of the transition matrix used by the full PageRank algorithm: $P = (1 - \beta)E + \beta P$. The matrix E is a 4×4-matrix in which all entries are $1/4$. For which value of β will the impartial web surfer spend one-third of his time visiting the pair (C, D)?

Problem 4. (a) Show that the $N \times N$-matrix C used in the discrete cosine transform for $N = 4$ is given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \gamma & \delta & -\delta & -\gamma \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \delta & -\gamma & \gamma & -\delta \end{pmatrix}$$

Express the two unknowns γ and δ in terms of the cosine function.
(b) Using the trigonometric identity $\cos 2\theta = 2\cos^2\theta - 1$, explicitly give the numbers γ and δ. (Here "explicitly" means as an algebraic expression with integer numbers and radicals but without the cosine function.) Using these expressions, show that the second line of C represents a vector with unit norm as is required by the orthogonality of C.

Problem 5. Show that for $N = 2^n$ there exists an orthogonal basis in $\mathbb{R}^N$ such that the vectors in this basis have coordinates either 1 or -1. Show that such a basis does not exists for $N = 3$. Hint for $N = 2^n$ (see also p.379, immediately after (12.7)):

(a) Consider N functions $f_\alpha = \prod_{j=1}^n \phi_{\alpha_j}(x_j)$, where $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \in \{0, 1\}$, are all sequences of $0, 1$ of length n, and $\phi_0(t) = \cos t, \phi_1(t) = \sin t$;
(b) Consider N points $p_\beta = (p_{\beta_1}, ..., p_{\beta_n})$, where $\beta = (\beta_1, ..., \beta_n)$, $\beta_j \in \{0, 1\}$ are all sequences of $0, 1$ of length n and $p_0 = \frac{\pi}{4}$ and $p_1 = \frac{3\pi}{4}$;
(c) Define $\mathbb{R}^N \ni e_j = ((f_j(p_{00...0}), ..., f_j(p_{11...1}))$. Show that this is (almost) what we need.

Problem 6. Beat patterns are a well-known musical phenomenon. When two instruments (physically close to each other) play nearly the same note at the same intensity, the perceived sound varies regularly in intensity with time. In other words, the perceived amplitude oscillates periodically. This oscillation may be slow (once every few seconds) or fast (several times per second).

(a) Two flutes emit sound waves f_1 and f_2 with frequencies ω_1 and ω_2 respectively:
$$f_1(t) = \sin(\omega_1 t) \quad \text{and} \quad f_2(t) = \sin(\omega_2 t).$$
(We neglect the harmonics, which we assume are weak.) The resulting sound is $f = f_1 + f_2$. Show that we can write f in the form
$$f(t) = 2\sin(\alpha t)\cos(\beta t),$$
and determine α and β in terms of ω_1 and ω_2.
(b) Suppose that ω_1 is a well-tempered E at 659.26 Hz, and that ω_2 is a true E at 660 Hz. Show that the ear would perceive f as a frequency close to these two, but with an amplitude varying with a period of about $\frac{4}{3}$ seconds. This is an example of a beat pattern.

Problem 7. We have shown that sampling every $\Delta = \frac{1}{44{,}100}$ seconds allows for all sounds in the (average) human audible spectrum to be reproduced. The problem is that musical instruments often produce harmonic frequencies above our hearing range with $N_{\max} = 20{,}000$ Hz. When the recording is sampled, a frequency $N > N_{\max}$ will be perceived as a sound with frequency between 0 and $N_{\max}$. (See

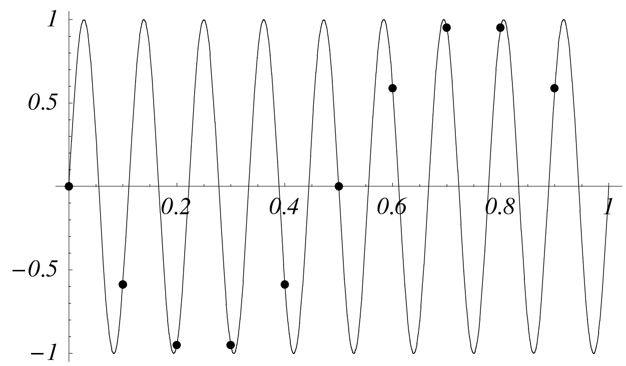

FIGURE 2. Problem 7

Figure 2, where the sampled points appear to describe a sinusoidal curve with a lower frequency than that which actually generated them.) This problem is known as aliasing, since certain frequencies are aliased (appear as) other frequencies after sampling. This problem appears in all domains in which signals are digitized. For example, it appears as countour banding or moire patterns in digital photography. This effect is closely related to another distortion commonly encountered in movies: a spoked wheel rotating quickly in one direction may appear to be rotating in the opposite direction. Determine the frequency N' that the frequency $N \geq N_{\max}$ will be aliased to after sampling. (Obviously, this aliased frequency must satisfy $0 < N' \leq N_{\max}$.)