ENGMATH2011
EXERCISE 4

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Abstract. These are some of the problems suggested to our class in high school.

1. Various problems

Problem 1.1. One end of a rubber band is fixed, and each second the second end of the band is pulled by 1 meter. A bug crawls with a constant speed 1cm/sec along the band, starting from the fixed end, when the band is 1 meter long. Will the bug get to the other end? If yes, how long it will take?

Problem 1.2 (Farey sequence and Ford circles). Two equal circles C_0 and C_1 touching each other and the real line at points 0 and 1. Let us generate infinite sequence of circles touching the real line as follows: as soon as there is a pair of circles tangent each other and the real line, add a circle touching these two circles and the real line (lying inside the curvilinear triangle formed by two arcs of the circles and the segment of the real line). We get an infinite sequence of circles touching the real line

(1) Compute the coordinates of the points of tangency (where the circles touch the real line.
(2) What are the radii of the circles.

Problem 1.3 (Pick formula). On square-ruled paper lies a polygon P with vertices in the grid points. Prove that the area of P is equal to $b + i/2 - 1$, where b is the number of grid points inside P and i is the number of grid points on the boundary of P.

Problem 1.4. (Continuation) Three grasshoppers sit at points of the integer grid on plane, at vertices of a triangle of area 1. They jump as follows: one of grasshoppers (call him A) jumps over another (call him B) in such a way that B sits exactly at the midpoint of the segment joining the old and the new position of A. Could it be that the triangle that they form after several jumps has integer points on sides (except vertices)? What triangles one can get this way?

Problem 1.5 (Mixed volume). For two convex polygons A, B on plane we can define their *Minkowski sum* as $A + B = \{a + b | a \in A, b \in B\}$. For a convex polygon A on plane and a real number $\lambda \in \mathbb{R}$ define $\lambda A = \{\lambda a | a \in A\}$.

(1) Will the set of convex polygon on plane form a linear space with respect to these operations?
(2) Prove that $S(\lambda A + \mu B) = \lambda^2 S(A) + 2\lambda\mu S(A, B) + \mu^2 S(B)$

Problem 1.6. Construct a bivariate polynomial $p(x, y)$ such that $p(x, y)$ never vanishes, but the $\inf_{x,y \in \mathbb{R}} p(x, y) = 0$.

Problem 1.7. Let the function $f(m,n)$ of two integer values has the following property:

$$4f(m,n) = f(m-1,n) + f(m,n-1) + f(m+1,n) + f(m,n+1) \tag{1.1}$$

and, moreover, $f(m,n) \geq 0$ for all m, n. Prove that this function is constant.

Problem 1.8. Union of n figures on plane has are n. Prove that for all $k < n$ there exist k figures with area of their union greater than k.

Problem 1.9. Prove that in a sequence of $mn+1$ different numbers there is either an increasing subsequence of length $m+1$ or decreasing subsequence of length $n+1$.

Problem 1.10. A polynomial $p(x)$ has degree 100, but only 7 non-zero coefficients. Prove that it has no more than 13 real roots.

Problem 1.11. For the polynomial $P(x) = a_0 x^n + ... + a_n$ the sum of its odd coefficients is equal to the sum of its even coefficients. Will it be true for the polynomial $P(x)(x^5 - 3x^3 + 3x + 1)$?