Handout1: Introduction to the PCP theorem

March 5 and 12, 2003

Language L is said to have a probabilistically checkable proof with parameters (r, q) if there is a randomized polynomial time algorithm V with the following properties:
Completeness. For every $x \in L$ there is a witness w such that $V^w(x)$ accepts (with probability 1). (This is known as *perfect completeness*.)
Soundness. For every $x \notin L$ and every w, $Pr[V^w(x) \text{ accepts}] \leq 1/2$.
$V^w(x)$ is *nonadaptive* and works in three phases. First $V(x)$ uses up to r random bits in order to list at most q bit locations. Then it queries w at these q locations and gets back the bit values. Finally, based on the values received, it accepts or rejects.

PCP theorem, version 1. $NP \subset PCP(\log n, 1)$. Namely, 3SAT has a PCP with $r = O(\log n)$ and $q = O(1)$.

PCP theorem, version 2. Max-3SAT is NP-hard to approximate within a ratio of ρ, for some $\rho < 1$. Namely, there is a polynomial time algorithm that receives as input a 3CNF formula ϕ and outputs a 3CNF formula ψ with the following properties.

- If ϕ is satisfiable, then ψ is satisfiable.
- If ϕ is not satisfiable, then ψ is at most ρ-satisfiable.

Homework (hand in by March 26). When proving hardness of approximation results for max-3SAT, we wanted q to be constant. For max-clique, we did not care. The following assignment aims to illustrate which of the two cases is more "typical".

A *vertex cover* in a graph is a set of vertices that touches all edges. Min-VC is the problem of finding a vertex cover of minimum cardinality. A close cousin of it is max-clique (the vertices not in the vertex cover form a clique in the edge-complement of the graph). Hence the reduction from PCP to max-clique should tell us something about the hardness of approximating min-VC. What hardness of approximation ratio for min-VC do we get as a function of q?