Random Walks on Regular and Irregular Graphs

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Abstract

For an undirected graph and an optimal cyclic list of all its vertices, the *cyclic cover time* is the expected time it takes a simple random walk to travel from vertex to vertex along the list, until it completes a full cycle. The main result of this paper is a characterization of the cyclic cover time in terms of simple and easy to compute graph properties. Namely, for any connected graph, the cyclic cover time is $\Theta(n^2 d_{ave}(d^{-1})_{ave})$, where n is the number of vertices in the graph, d_{ave} is the average degree of its vertices, and $(d^{-1})_{ave}$ is the average of the inverse of the degree of its vertices. Other results obtained in the processes of proving the main theorem are a similar characterization of minimum resistance spanning trees of graphs, improved bounds on the cover time of graphs, and a simplified proof that the maximum commute time in any connected graph is at most $4n^3/27 + o(n^3)$.

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1 Introduction

Let $G = G(V, E)$ be a simple connected graph on n vertices and m edges. For any vertex $v \in V$, d_v denotes the degree of v (the number of edges incident with v). We consider random walks on G, where at each step the random walk moves to a vertex chosen at random with uniform probability from the neighbors of the current vertex. For two vertices $u, v \in V$, the *hitting time* $H[u, v]$ is the expected number of steps it takes a walk that starts at u to reach v, and the *commute time* $C[u, v]$ is the expected number of steps that it takes a walk to go from u to v and back to u (that is, $C[u, v] = H[u, v] + H[v, u]$). The *cover time* of a graph, $EC[G]$, is the expected number of steps it takes a random walk to visit all vertices of the graph, starting at the worst possible vertex (that maximizes this value).

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Aleliunas *et al.* [2] showed that for any connected graph, $EC[G] < 2nm$. This bound has been refined by Kahn *et al.* [14], who proved a bound of $EC[G] \leq 4n^2 d_{ave}/d_{min}$, where d_{ave} is the average degree of the graph, and d_{min} is its minimum degree. This bound takes into account the structure of the graph: for regular graphs the bound is low, $O(n^2)$, whereas for irregular graphs, those which have a high ratio d_{ave}/d_{min}, the bound is higher.

In this paper we further study the relation between "regularity" of a graph and random walks. However, we use a different measure of regularity, namely $d_{ave}(d^{-1})_{ave}$, where $(d^{-1})_{ave}$ is the average of the inverse of the degrees of the vertices. This measure obtains its minimum 1 on regular graphs, and its maximum $\Omega(n)$ on highly irregular graphs that have a linear number of vertices of constant degree and a linear number of vertices of degree $\Omega(n)$. It was argued in [10] that this measure is preferable to d_{ave}/d_{min}, since it is more robust. Introduction of even a single vertex of small degree can cause d_{ave}/d_{min} to increase by a multiplicative factor of $\Theta(n)$, whereas $d_{ave}(d^{-1})_{ave}$ would increase by at most a constant factor.

Using our regularity measure, we provide an almost tight characterization of a property that is closely related to the cover time. The *cyclic cover time*, $ECC[G]$, is the expected number of steps it takes to visit all vertices of the graph in a prespecified cyclic order, for the best cyclic order (that minimizes this value). That is,

$$ECC[G] = H[v_{i_1}, v_{i_2}] + H[v_{i_2}, v_{i_3}] + ... + H[v_{i_{n-1}}, v_{i_n}] + H[v_{i_n}, v_{i_1}]$$

where $(i_1, i_2, ..., i_n)$ is a permutation that minimizes the above sum. Clearly, $ECC[G] > EC[G]$.

Our main theorem states that for any graph G, $ECC[G]$ is characterized up to a constant factor by $d_{ave}(d^{-1})_{ave}$.

Theorem 1 *For any connected graph G,*

$$(\sum_{v \in V} d_v)(\sum_{v \in V} \frac{1}{d_v + 1}) \leq ECC[G] \leq \frac{10}{3}(\sum_{v \in V} d_v)(\sum_{v \in V} \frac{1}{d_v + 1})$$

Observe that $\sum_{v \in V} d_v = 2m = nd_{ave}$, and $\sum_{v \in V} 1/d_v = n(d^{-1})_{ave}$. Thus the features of the graph that characterize its cyclic cover time can be abstracted to the single easily computable quantity $d_{ave}(d^{-1})_{ave}$ (in particular, it is computable in deterministic logarithmic space), with loss of only a constant factor in the accuracy of the value computed for the cyclic cover time.

A simple corollary of Theorem 1 is that for any connected graph, $EC[G] < \frac{10}{3} n^2 d_{ave}(d^{-1})_{ave}$. This is an improvement of the [14] bound on the cover time, since for every graph, $d_{ave}(d^{-1})_{ave} \leq d_{ave}/d_{min}$. However, the bounds that we obtain on the cover time are far from tight. Observe that the bounds that we derive are $\Omega(n^2)$, whereas the cover time of a graph may be as low as $O(n \log n)$ (for a clique, or a star). Better bounds on the cover time can sometimes be obtained from the relation $EC[G] \leq \max_{u,v}[H[u,v]] \ln n$ [15]. The maximum hitting time on a graph can be computed in polynomial time. Alternatively, it can be bounded from above by the maximum commute time, which in turn can be bounded by $2mD$, where D is the diameter of the graph, or even characterized exactly by $2mR_{\text{eff}}$, where R_{eff} is the maximum *effective resistance* in the graph (see [6], and also Section 3). We remark that it is not known whether the diameter of a graph or its maximum effective resistance are computable in deterministic logspace.

2 A travelling salesperson formulation of cyclic cover time

For any undirected connected graph $G = G(V, E)$, we can construct the following directed weighted complete graph $G' = G'(V, W)$, where W is the sequence of weights associated with the edges of G'. The vertex set of G' is identical to that of G. Any two vertices $u, v \in V$ are connected by two antiparallel directed edges in G'. The weight of edge (u, v) is set to be identical to the hitting time $H[u, v]$. The problem of computing the cyclic cover time of G is thus equivalent to solving *travelling salesperson* (TSP, finding the Hamiltonian cycle of least weight) on the directed graph G'. (One may regard the problem of finding the optimal cyclic cover tour as a *drunken salesperson* problem, where the salesperson wanders at random from town to town, searching for the next town on his/her list. What is the optimal ordering of towns on the list?)

The directed graph G' can be made into an undirected graph, which we also denote by G' (so as not to introduce additional notation). For any pair of vertices $u, v \in V$, the two antiparallel edges are replaced by a single edge with weight equal to the commute time $C[u, v]$. Let $TSP(G')$ denote the weight of the minimum weight Hamiltonian cycle in the undirected graph G'. Then $TSP(G')$ is equal to exactly twice the cyclic cover time on G, by the well known fact that along any cycle, the sum of hitting times going along one direction is equal to the sum of hitting times going along the opposite direction (see e.g. [8]).

In order to obtain an upper bound on the cyclic cover time, we shall use the following relation between spanning trees and Hamiltonian cycles. Let $MST(G')$ denote the weight of the minimum weight spanning tree of G'. Clearly, $MST(G') \leq TSP(G')$. Furthermore, if the weights satisfy the triangle inequality, then it is a simple matter to show that $TSP(G') \leq 2MST(G')$ (see e.g. [13]). Thus in our case of cyclic cover time, it remains to bound $MST(G')$.

Remark: We do not know if exact computation of the cyclic cover time is NP-hard. The algorithm of Christofides [7, 13] approximates *metric* TSP within a factor of 3/2, and can be applied to approximate the cyclic cover time, as the commute time (weight of the edges in G') is computable in polynomial time, and satisfies the triangle inequality (that is, $C[u,w] \leq C[u,v] + C[v,w]$). By [3], there is some constant ϵ such that it is NP-hard to approximate metric TSP to within a ratio of $1+\epsilon$. We do not know whether the extra structure of our graph G' can be exploited to improve over the approximation ratio obtainable for metric TSP.

3 Random walks and electrical resistance

There is a well known correspondence between random walks and resistance of electrical networks [9, 6, 16]. View each edge of G as a resistor of 1 ohm. The *effective resistance* between vertices u and v, denoted by $R[u,v]$, is the voltage that develops at u if a current of 1 amp is injected into u, and v is grounded. In [6] it is shown that the effective resistance captures the commute time. Namely, for any connected graph with m edges, and any two vertices u and v,

$$C[u,v] = 2mR[u,v]$$

We shall use three properties about effective resistance:

1. *Serial connection:* resistors that are connected in series can be replaced by a single resistor whose resistance is the sum of the resistances.
2. *Parallel connection:* resistors that are connected in parallel can be replaced by a single resistor whose conductance (the inverse of resistance) is the sum of the conductances.
3. *Shortcut principle:* the effective resistance between any two vertices in a network does not increase if we "short cut" (connect by a resistor of resistance 0) vertices in the network.

The above three properties imply the following proposition:

Proposition 2 *The effective resistance between any two vertices $u, v \in V$ satisfies:*

$$R[u,v] \geq \frac{1}{d_u+1} + \frac{1}{d_v+1}$$

Proof: We first show that if u and v are not connected by an edge in G (that is, $(u,v) \notin E$), then $R[u,v] \geq \frac{1}{d_u} + \frac{1}{d_v}$. Collapse all vertices of $V \setminus \{u,v\}$ to a single vertex w, obtaining a graph GS. By the shortcut principle, $R_{GS}[u,v] \leq R[u,v]$. By parallel connection, $R_{GS}[u,w] = 1/d_u$. Likewise, $R_{GS}[w,v] = 1/d_v$. By serial connection, $R_{GS}[u,v] = R_{GS}[u,w] + R_{GS}[w,v] = 1/d_u + 1/d_v$.

We now consider the case that $(u,v) \in E$. Construct from G a graph GW by removing the edge (u,v), and adding two new vertices w_1 and w_2, together with the four edges (u,w_1), (w_1,v), (u,w_2), (w_2,v). Observe that $R_{GW}[u,v] = R[u,v]$, since the pair of edges $[(u,w_i),(w_i,v)]$ (for $i=1$, or for $i=2$) can be replaced by a single 2 ohm resistor (serial connection principle), and thereafter the two 2 ohm parallel resistors can be replaced by a single 1 ohm resistor (parallel connection principle), giving back the original edge (u,v). In GW, there is no edge (u,v), and the degree of each of the vertices u and v increased by 1. Continue as above. $\Box$

As a corollary we obtain the easy direction of Theorem 1.

Corollary 3 *For any connected graph, the cyclic cover time satisfies*

$$ECC \geq (\sum_{v \in V} d_v)(\sum_{v \in V} \frac{1}{d_v+1})$$

Proof: Consider an arbitrary cyclic ordering of the vertices of G. The sum of resistances along this cyclic order is at least $2\sum_{v \in V} \frac{1}{d_v+1}$. Hence the sum of commute times is at least $2m \cdot 2\sum_{v \in V} \frac{1}{d_v+1}$, and the cyclic cover time is at least $2m\sum_{v \in V} \frac{1}{d_v+1}$. $\Box$

The above bound is tight for complete graphs.

In order to obtain an upper bound on the cyclic cover time, or $TSP(G')$, we shall bound $MST(G')$. For reasons of convenience, the weights assigned to edges of G' are the effective resistances $R[u,v]$ (rather than the commute times $C[u,v]$). It will turn out that for the proof of Theorem 1, there is no need to consider edges of G' that were not edges in the original graph G, and they are assigned “infinite weight”. Our goal is to upper bound R_{span}, the weight of the spanning tree of minimum weight (resistance). By previous discussion it follows that the cyclic cover time of G is at most $2mR_{span}$.

The approach of using R_{span} in order to bound the cover time originates from [2, 14]. Our current work was motivated by the conjecture in [10] that $R_{span} = \Theta(\sum_{v \in V} 1/d_v)$.

4 The excess resistance lemma

Definition: The *excess resistance* $\delta[(u,v)]$ of edge $(u,v) \in E$ is defined by:

$$\delta[(u,v)] = R[u,v] - (\frac{1}{d_u+1} + \frac{1}{d_v+1})$$

Observe that by Proposition 2, $\delta[(u,v)] \geq 0$, with equality if and only if $d_u = d_v$ and there are $d_u - 1$ intermediate vertices that are adjacent both to u and to v (in addition to the one edge (u,v)).

The following identity (the *excess resistance lemma*) plays a fundamental role in our paper:

Lemma 4 *For any graph $G = G(V, E)$ with c connected components, the sum of excess resistances along its edges satisfies:*

$$\sum_{(u,v)\in E} \delta[(u,v)] \;=\; \sum_{v\in V} \frac{1}{d_v + 1} \;-\; c$$

Proof: We present without proof an identity due to Foster [12, 16]:

Lemma 5 *Let $G = G(V, E)$ be a graph with n vertices and c connected components. The sum of effective resistances along the edges of G satisfies:*

$$\sum_{(u,v)\in E} R[u,v] \;=\; n - c$$

We return to the proof of Lemma 4. By definition of the excess resistance:

$$\begin{aligned}
\sum_{(u,v)\in E} R[u,v] \;&=\; \sum_{(u,v)\in E} \left(\frac{1}{d_u + 1} + \frac{1}{d_v + 1} + \delta[(u,v)]\right) \\
&=\; \sum_{v} \frac{d_v}{d_v + 1} + \sum_{(u,v)\in E} \delta[(u,v)]
\end{aligned}$$

The excess resistance identity follows from Foster's identity. $\square$

5 Bounds on the commute time

The following theorem improves the leading constant (by a factor of 3) over a similar theorem proved in [10]. This results in improved time estimates for deciding undirected s-t-connectivity by space-efficient randomized algorithms [2, 10].

Theorem 6 *For any pair of vertices s and t in a connected graph, the commute time satisfies:*

$$C[s,t] < 6m \sum_{v\in V} \frac{1}{d_v + 1}$$

Proof: Consider any simple path $p = s, v_1, v_2, ..., v_\ell, t$, connecting s and t (not necessarily the shortest path). We use the excess resistance lemma to bound the resistance along this path.

$$R[p] \;=\; R[s, v_1] + R[v_1, v_2] + ... + R[v_\ell, t]$$

$$= \frac{1}{d_s+1} + \frac{1}{d_{v_1}+1} + \delta[(s,v_1)] + \ldots + \frac{1}{d_{v_\ell}+1} + \frac{1}{d_t+1} + \delta[(v_\ell,t)]$$

$$< 2\sum_{v\in V} \frac{1}{d_v+1} + \sum_{(u,v)\in E} \delta[(u,v)] \; < \; 3\sum_{v\in V} \frac{1}{d_v+1}$$

The sum of commute times along path p is an upper bound on $C[s,t]$, and it satisfies:

$$C[p] \;=\; 2mR[p] \;<\; 6m\sum_{v\in V} \frac{1}{d_v+1}$$

□

The above bound on the maximum commute time in a graph is best possible (up to low order term). This can be verified by considering specific examples, such as a path of n edges (and $n+1$ vertices), for which the commute time between the two endpoints is $2n^2$. A more striking demonstration of the tightness of Theorem 6 is provided by the following corollary:

Corollary 7 *For any two vertices s and t in a connected graph of n vertices, the commute time satisfies:*

$$C[s,t] \leq \frac{4}{27}n^3 + o(n^3)$$

Proof: We first transform G into a new graph G' for which almost all vertices have degree at least 2. Repeatedly delete vertices of degree 1 (except for s and t), and re-append them as a path leading out of t. This does not change the number of edges in G, nor the effective resistance between s and t. Hence $C_{G'}[s,t] = C[s,t]$. Note that in G' at most two vertices have degree 1 (s, and the vertex at the end of the path leading out of t).

Assume now that G' has k vertices of degree at least $\sqrt{n}$, and $n-k$ vertices of degree less than $\sqrt{n}$. Then the number of edges in G is at most $\frac{k^2}{2} + (n-k)\sqrt{n}$, and the expression $\sum_{v\in V} \frac{1}{d_v+1}$ is at most $\frac{n-k}{3} + \frac{k}{\sqrt{n}} + 1$. The product of these two expressions is maximized when $k \simeq 2n/3$, and the bound on the commute time follows. □

The above corollary is not new. Brightwell and Winkler [5] show that the maximum hitting time in a graph is at most $\frac{4}{27}n^3 + o(n^3)$, and in fact they find the graph for which the hitting time is maximized (the *lollipop* graph). A similar bound on the commute time is implicit in [8]. The same upper bound also holds for the cyclic cover time, as proven in [11].

6 Regular graphs

We now turn to bound R_{span}, the resistance of the spanning tree of minimum resistance. We start with the case of d-regular graphs. For d-regular graphs, Kahn *et al.* [14] prove that the cover time is bounded by $4n^2$. Their proof actually shows that $R_{span} \leq 4n/d$, and hence the cyclic cover time is bounded by $4n^2$. We improve upon the bounds of [14].

Theorem 8 *In connected d-regular graphs, the resistance of any spanning tree T satisfies:*

$$\frac{2(n-1)}{d+1} \leq R[T] < \frac{3(n-1)}{d+1}$$

Proof: The lower bound on $R[T]$ is a trivial consequence of proposition 2. The upper bound is by application of Lemma 4. $\square$

Thus all spanning trees in a regular graph have roughly the same resistance. The *clique* demonstrates that there are d-regular graphs for which all spanning trees have the minimum possible value. The *necklace* (a cycle of cliques, where an edge is removed from each clique so as to allow the connection of adjacent cliques without violating regularity) demonstrates that there are d-regular graphs for which all spanning trees have nearly the maximum possible value. The following graph demonstrates that it is possible to obtain a ratio of nearly $3/2$ between the minimum and maximum resistance spanning trees (for large d, and $n >> d$). Consider $n/(d+1)$ disjoint cliques, each of size $d+1$. In each clique, remove a cycle (of $(d+1)$ edges). Now create a cycle of cliques, where each clique is connected to each neighboring clique by $(d+1)$ *connecting* edges. It can be verified that the resistance of clique edges is roughly $2/d$, whereas the resistance of connecting edges is roughly $3/d$. Hence a spanning tree that is based mainly on connecting edges will have resistance $3/2$ times as high as a spanning tree that is based mostly on clique edges.

7 Proof of main theorem

Theorem 8 can be generalized to arbitrary graphs, showing that the resistance of any spanning tree satisfies

$$\frac{n-2}{d_{max}+1} + \sum_v \frac{1}{d_v+1} \leq R[T] < \frac{n-1}{d_{min}+1} + 2\sum_v \frac{1}{d_v+1}$$

For the purpose of proving Theorem 1, we need an upper bound on R_{span} that does not depend on d_{min}. Our problems would be solved if we could replace $\frac{n-1}{d+1}$ in Theorem 8 with $\sum_{v\in V} \frac{1}{d_v+1}$. However, the resulting proposition would be false.

The lower bound of Theorem 8 on $R[T]$ would be off by a factor of roughly 2, as can be seen by considering a bipartite graph $G_{a,b}$, with a left hand side vertices, b right hand side vertices, and $b >> a$. The effective resistance along any edge is $\frac{a+b-1}{ab}$ (by symmetry, and Foster's identity), and hence all spanning trees have resistance roughly b/a (assuming $b >> a$). For $G_{a,b}$, $\sum_{v \in V} \frac{1}{d_v+1} \simeq b/a$. Hence $R[T] \simeq \sum_{v \in V} \frac{1}{d_v+1}$.

The upper bound of Theorem 8 on $R[T]$ is violated to a more dramatic extent. Consider the graph composed of a clique on k vertices, with $\sqrt{k}$ additional vertices, each connected via *connecting edges* to $\sqrt{k}$ distinct vertices in the clique. For this graph, $\sum_{v \in V} \frac{1}{d_v+1} \simeq 2$, whereas the maximal resistance spanning tree (using all connecting edges) has resistance roughly $\sqrt{k}$.

Hence, unlike the case for regular graphs, it is not true in general that for any spanning tree T, $R[T] = O(\sum_{v \in V} \frac{1}{d_v+1})$. However, the following theorem shows that for any connected graph, the average resistance of a spanning tree (when the spanning tree is chosen at random) is $O(\sum_{v \in V} \frac{1}{d_v+1})$.

Theorem 9 *For any connected graph G, the expected resistance of the spanning trees of G satisfies:*

$$\mathcal{E}_T[R[T]] \leq \frac{10}{3} \sum_{v \in V} \frac{1}{d_v + 1}$$

Proof: The fraction of spanning trees that use edge (u, v) is exactly $R[(u, v)]$ [4]. Hence:

$$\mathcal{E}_T[R[T]] = \sum_{(u,v) \in E} (R[(u, v)])^2 = \sum_{(u,v) \in E} (\frac{1}{d_u + 1} + \frac{1}{d_v + 1} + \delta[(u, v)])^2$$

To bound the above sum we use the following technical lemma:

Lemma 10 *For any edge (u, v),*

$$(\frac{1}{d_u + 1} + \frac{1}{d_v + 1} + \delta[(u, v)])^2 \leq \frac{2}{d_u(d_u + 1)} + \frac{2}{d_v(d_v + 1)} + \frac{4}{3}\delta[(u, v)]$$

Proof: For any integer positive values of d_v and d_u, the above inequality holds for the minimum possible value $\delta = 0$, and maximum value $\delta = 1 - \frac{1}{d_u+1} - \frac{1}{d_v+1}$ (calculations omitted). Since the left hand side of the inequality is quadratic in δ, and the right hand side is linear in δ, the inequality must also hold for all intermediate values of δ. $\square$

Hence:

$$\mathcal{E}_T[R[T]] \leq \sum_{(u,v) \in E} (\frac{2}{d_u(d_u + 1)} + \frac{2}{d_v(d_v + 1)} + \frac{4}{3}\delta[(u, v)])$$

$$\leq 2 \sum_{v \in V} \frac{d_v}{d_v(d_v + 1)} + \frac{4}{3} \sum_{(u,v) \in E} \delta[(u, v)] \ \leq \ \frac{10}{3} \frac{1}{d_v + 1}$$

□

This completes the proof of Theorem 1. Observe that if we could improve the leading constant in Theorem 1 from 10/3 to 3, this would imply an alternative proof to $ECC \leq \frac{4n^3}{27} + o(n^3)$ [11], along the lines of Corollary 7. One might hope to achieve this by bounding the minimum spanning tree, or R_{span}, rather than the average resistance of random spanning trees. However, it is not true that for any graph $R_{span} \leq 3\sum_{v\in V}\frac{1}{d_v+1}$. Consider a full binary tree of depth D (having $2^{D+1}-1$ vertices). Replace every leaf by a k-clique, with $k >> d$. It can be easily verified that $\sum_{v\in V}\frac{1}{d_v+1} \simeq \frac{5}{4}\cdot 2^D$, and that for any spanning tree T, $R[T] \simeq 4\cdot 2^D$. Hence $R_{span} \simeq \frac{16}{5}\sum_{v\in V}\frac{1}{d_v+1}$.

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