Algorithms – handout 3

Dynamic programming and bounded tree width

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We shall touch upon an advanced topic – the theory of "Graph Minors" by Robertson and Seymour. As a motivating example, we show that maximum weight independent set on trees can be solved in polynomial time, using dynamic programming. (Note however that bandwidth is NP-hard on trees.) We then introduce the notion of a tree decomposition of a graph.

Definition: A *tree decomposition* of a graph $G(V, E)$ is a tree T, where each vertex i of T is labeled by a subset $H_i \subset V$ of vertices of G, each edge of G is in a subgraph induced by at least one of the H_i, and the vertices of T labeled by any vertex $v \in V$ are connected in T. The *tree width* of G is the minimum integer p such that there exists a tree decomposition G with all subsets of cardinality at most $p + 1$.

We shall show that a tree has tree width 1, a series-parallel graph has tree-width 2, a k-clique has tree width $k - 1$, and an n-vertex square mesh has tree width $\Theta(\sqrt{n})$.

Definition: H is a *minor* of G if it can be obtained from G by a sequence of operations of taking subgraphs and edge *contractions* (merging endponts together).

Kuratowski showed that non-planar graphs must contain either K_5 or $K_{3,3}$ as minors.

The following theorems we do not prove:

Theorem: Let H be a planar graph. If G has no H-minor, then the tree width of G is bounded by some function of H, independent of G.

Theorem: For graphs that have bounded tree width, a corresponding tree decomposition can be found in polynomial (in fact, linear) time.

We shall use this and dynamic programming to design a polynomial time algorithm for k-coloring graphs of bounded tree width.

Robertson and Seymour show that every family of graphs that is closed w.r.t. taking minors has a finite *obstruction set* of minors (such as K_5 or $K_{3,3}$ for planarity). They further show that for every fixed H, testing whether G contains H as a minor can be done in time $O(n^3)$. A consequence of their theory is that any property of graphs that is inherited by minors (such as being planar except for k vertices, linkless embedding in 3-dimensional space, etc.) can be *decided* in polynomial time. (Their theory does not necessarily produce an algorithm, and if it does, the hidden constants in the running time are often huge.)

Homework (easy):

1. Show that if H is not planar, then G not containing H as a minor may have arbitrarily large tree width.

2. Show that the tree width of a graph is at least as large as that of any of its minors.