1 Lecture 9, December 16, 1999

For certain classes of optimization problems, we have a characterization of the conditions under which the naive greedy algorithm produces the optimal solution. *Matroids* are a well known characterization of this form.

1.1 Matroids

Let $S = \{e_1, \ldots, e_n\}$ be a finite set, and let $F \subset 2^S$ be a collection of subsets of S, called *independent* sets. We say that F is *hereditary* if it is closed under taking subsets. That is, if $X \in F$ and $Y \subset X$, then $Y \in F$. For example, if S is the set of vertices in a graph and F is the collection of all independent sets in the graph, then F is hereditary.

Let $c : S \to R^+$ be a cost function on the elements of S. We wish to find $X \in F$ that maximizes $\sum_{e \in X} c(e)$. A greedy algorithm may construct X as the final set in a sequence $X_0, X_1, \ldots$ as follows:

1. Let $X_0 = \phi$.
2. Let $X_{i+1} = X_i \bigcup \{e_j\}$, where e_j is an element in $S \setminus X_i$ with maximum value of $c(x_j)$.

(A more efficient implementation will first sort all elements e in order of decreasing weight $c(e)$.)

For the hereditary family of independent sets in graphs, this greedy algorithm will fail to find a maximum independent set. (For example, consider a star in which the middle vertex has slightly higher cost then any of the other vertices.) But there are hereditary families for which the greedy algorithm produces the optimal solution, regardless of the particular cost function c. These are known as *matroids.*

Definition 1 *A hereditary family F of sets is a* matroid *if for all $X, Y \in F$, if $|X| > |Y|$ then there is some $x \in X \setminus Y$ such that $Y \bigcup \{x\} \in F$.*

The maximal independent sets of a matroid are called *bases.*

Proposition 1 *All bases of a matroid have the same cardinality, called the* rank *of the matroid.*

Proof: By the matroid property, an indendent set in F is maximal if and only if there is no set of larger cardinality in F. $\Box$

Remark: the terminology for matroids comes from terminology for matrices. Consider all columns of a matrix. Then we can define the following matroid. Independent sets correspond to sets of linearly independent columns. Bases correspond to those sets of columns that form a basis for the column subspace of the matrix. The rank of the matrix is the rank of the associated matroid.

Theorem 2 *The naive greedy algorithm optimizes over a hereditary family F for every cost function c iff F is a matroid.*

Proof: Assume that F is not a matroid, and let $X, Y \in F$ be such that $|X| > |Y|$ and for every $x \in X \setminus Y$, $Y \bigcup \{x\} \notin F$. Define a cost function c that is negligible on $S \setminus (X \bigcup Y)$, equal to $1+1/|X|$ for each $e \in Y$, and equal to 1 on the remaining members of X. Then the greedy algorithm will choose Y, for a total cost of $c(Y)$. But choosing X is strictly better, by the inequalities below:

$$c(Y) = |Y|(1 + \frac{1}{|X|}) = |Y| + \frac{|Y|}{|X|} < |Y| + 1 \leq |X| \leq c(X).$$

Assume now that F is a matroid. Observe that the maximum X must be a basis, and that the greedy algorithm must also produce a basis. Let $e_1, e_2, \ldots e_k$ be the elements chosen by the greedy algorithm. Let $f_1, f_2, \ldots f_l$ be the elements of any arbitrary basis, and assume that they are sorted in descending order. Then necessarily $c(e_i) \geq c(f_i)$ for every i. For the sake of contradiction, assume otherwise. Let j be the least index with $c(f_j) > c(e_j)$. Then also $c(f_i) > c(e_j)$ for all $i < j$. Let $X = \{f_1, \ldots, f_j\}$ and $Y = \{e_1, \ldots e_{j-1}\}$. Then $X, Y \in F$ and $|X| > |Y|$. Hence Y can be extended by some member of $X \setminus Y$, which is a more greedy choice than e_j. This contradicts the assumption that $e_1, e_2, \ldots e_k$ is the output of the greedy algorithm. $\square$

A well known example of matroids is that of the forests of a graph. For a graph $G(V, E)$, the set S is E, and $X \in F$ if the edges of X do not close a cycle. It is easy to see that F is hereditary. To see that the matroid property holds, observe that the number of connected components in the graph (V, X) is exactly $|V| - |X|$. Hence if $|X| > |Y|$, there are less connected components in (V, X) then in (V, Y), and hence at least one edge in X connects different connected components of Y, and hence does not close a cycle in (V, Y).

If G is connected, then the bases of the forest matroid are the spanning trees of G, and the greedy algorithm above produces a maximum weight spanning tree. To find a minimum weight spanning tree, the similar algorithm can be run in which edges are sorted in order of increasing rather than decreasing cost. Analogous to the proof of theorem 2 and using its notation, we will have that $c(e_i) \leq c(f_i)$ for every i. This algorithm for finding minimum spanning trees is known as Kruskal's algorithm, though it was already described by Borouvka in the 1920's.

1.2 Matroid duality

We first prove a strengthening of Proposition 1. For a matroid (S, F) and $S' \subset S$, let (S', F) be the hereditary family containing sets X' that are of the form $X \bigcap S'$ for $X \in F$.

Lemma 3 *A hereditary family (S, F) is a matroid iff for every $S' \subset S$, the maximal elements of (S', F) all have the same cardinality.*

Proof: Let F be a matroid, let $X', Y' \subset S'$ be $X \bigcap S'$ and $Y \bigcap S'$ for some $X, Y \in F$. Then $X', Y' \in F$ because F is hereditary. Assume that X' and Y' are both maximal in S'. Then neither can be extended by an element of the other, and the matroid property for (S, F) implies that they must be of the same cardinality.

Assume now that F is not a matroid, and let $X, Y \in F$ be such that $|Y| < |X|$ but Y cannot be extended by any element of X. Take $S' = X \bigcup Y$. Then Y is maximal in (S', F), but (S', F) contains also sets of larger cardinality (such as X). $\square$

Definition 2 *Given a matroid (S, F), its* dual *is (S, F^*), where*

$$F^* = \{X \subset S | S \setminus X \text{ contains a basis of } F\}$$

Theorem 4 *The dual of a matroid is a matroid. Furthermore, $(F^*)^* = F$.*

Proof: It is clear that (S, F^*) is hereditary. We shall show the matroid property.

Let (S, F) be a matroid. Let X, Y be in F^* and assume $|X| > |Y|$. We need to prove that Y can be extended by an element of X and still remain in F^*. Let r be the rank of (S, F). Let $S' = S \setminus (X \bigcup Y)$. Then (S', F) is a matroid. Let r' its rank. Then $r' \leq r$. If $r' = r$ we can move any element x from $X \setminus Y$ to Y and keep $Y \bigcup \{x\} \in F^*$. Hence assume that $r' < r$.

Let B_X be a basis of (S, F) that does not intersect X and let B_Y be a basis of (S, F) that does not intersect Y. By the definition of F^*, such bases exist.

Observe that necessarily $r' \geq r - |Y \setminus X|$, because $B_X \setminus (Y \setminus X) \in (S', F)$. Now let B' be a basis of (S', F) (or cardinality r'). It can be extended to a basis B of (S, F) using only elements from $X \setminus Y$, because $(S' \bigcup (X \setminus Y), F)$ has B_Y as a basis, and hence contains bases for (S, F). But $|X \setminus Y| > |Y \setminus X|$, because $|X| > |Y|$. Hence at least one element x of $X \setminus Y$ is not in B. Move this x to Y. This completes the proof that (S, F^*) is a matroid.

To see that $(F^*)^* = F$, observe that the bases of F^* are exactly the complements of the bases of F. Hence the bases of $(F^*)^*$ are the complements of the bases of F^*, which are the bases of F. As the bases of a matroid completely determine the rest of the independent sets (by the hereditary property), we are done. $\square$

1.3 Duality and planar graphs

A planar graph is a graph that can be drawn on the plane with no intersecting edges. For the following discussion we consider connected planar graph that need not be simple – they may have self loops and parallel edges. A planar embedding of a planar graph $G(V, E)$ divides the plane in a natural way into regions, called *faces*. A *geometric dual* G^* of a planar graph G is the graph obtained by placing a dual vertex in each face (including the outer face), and connecting two dual vertices if their respective faces share an edge. Hence there is a one-to-one correspondence between edges of G and G^*. A planar graph may have several different dual graphs, because it may have several different planar embeddings. We note that G^* is planar, and that G is a dual of G^*.

Proposition 5 *A set of edges T is a simple cycle in G iff it is a minimal cut (a cut is a set of edges whose removal disconnects a graph) in G^*.*

Consider now the forest matroid on a planar graph G. Its dual matroid is exactly the forest matroid on G^*. As spanning trees are the bases of these matroids, and a spanning tree for n vertices has $n - 1$ edges, we have Euler's formula for planar graphs:

$$(|V| - 1) + (f - 1) = |E|$$

where f denotes the number of faces.

Corollary 6 *Every simple planar graph (with no self loops or parallel edges) has a vertex of degree at most 5.*

Proof: W.l.o.g. we may assume that the graph is connected, as a disconnected graph can be connected by only adding edges and hence without decreasing any of the degrees. Similarly, we may assume that the graph is *triangulated*, meaning that each face is composed of exactly three edges, because if we have a face with more edges, we may add an edge to the graph (a chord) without decreasing any of the degrees. In a triangulated planar graph, $3f = 2|E|$ (by counting face-edge pairs in two different ways). Euler's formula then gives $|E| = 3|V| - 6$. Hence the average degree is $6 - 12/|V|$, implying a vertex of degree at most five. □

We previously used corollary 6 in the inductive coloring algorithm for planar graphs.

1.4 Intersection of matroids

Let (S, F_1) and (S, F_2) be matroids defined on the same set S. Then the interestion of the two matroids $(S, F_1 \bigcap F_2)$ are those sets $X \in F_1 \bigcap F_2$. We can define intersections of three matroids in a similar way.

Consider a bipartite graph $G(V, E)$ with two sides, V_1 and V_2. Define a matroid (S, F_1) with $S = E$ and $X \in F_1$ if X does not contain two edges that share the same end point in V_1. It is easy to check that F_1 is a matroid. Analogously, define F_2 such that $X \in F_2$ if X does not contain two edges that share the same end point in V_2. Then $(S, F_1 \bigcap F_2)$ is the set of matchings in the bipartite graph G. There are other well known combinatorial structures that can be represented as intersections of matroids.

The intersection of two matroids is not necessarily a matroid. Consider a bipartite graph with is just a path with four vertices. The middle edge is a matching that cannot be extended by any of the outer two edges, even though they form a larger matching. Hence the greedy algorithm is not guaranteed to produce maximum elements of $F_1 \bigcap F_2$. However, there are polynomial time algorithms for computing maximum matchings. This suggests that there is a more general algorithm of optimizing over intersections of two matroids, and indeed, there is such an algorithm.

The intersection of three matroids includes three dimensional matching, in which there are three sets V_1, V_2, V_3, hyperedges e that contain one element from each set, and a matching X includes sets of hyperedges such that no two of which share an endpoint (neither in V_1, nor in V_2, nor in V_3). It is NP-hard to compute a three dimensional matching of maximum cardinality, which shows that we cannot efficiently optimize over intersections of three matroids even when the cost function is uniform (unless P=NP).

References: *Handbook of Combinatorics*, edited by R. Graham, M. Grotschel and L. Lovasz, MIT Press and Elsevier Science B. V., 1995, contains several chapters on matroid theory, as well as on other topics of this course.