Handout 6 – Applications of LP duality

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We will use LP duality to prove the minimax theorem for two person zero sum games. (Historically, the minimax theorem was proved before LP duality.) We will comment on the relevance of this theorem to proving lower bounds for randomized algorithms.

We shall then use LP duality to revisit vertex cover on bipartite graphs.

Vertex cover (the complement of independent set) can be formulated as:
minimize $\sum_{i=1}^{n} v_i$ *subject to*:
$v_i + v_j \geq 1$, for each edge (v_i, v_j),
$v_i \in \{0, 1\}$, for each vertex v_i.
NP-hardness of vertex cover implies that integer programming is NP-hard.

We relax the integer program to a linear program by replacing the $v_i \in \{0, 1\}$ constraints by nonnegativity constraints $v_i \geq 0$. The gap between the value of the optimal solution for integer program and its relaxed fractional program is known as the *integrality gap*. For the case of vertex cover, it is a multiplicative factor of at most 2.

By studying the dual of this linear program we will derive a polynomial time algorithm for vertex cover in bipartite graphs. In particular we will show that if all coefficients of a linear program are integer and the constraint matrix A is *totally unimodular*, then the linear program (if feasible) has an integer optimal solution.

We shall also use duality and total unimodularity to prove the min-cut/max-flow theorem.

Homework

1. We saw in class that the LP that we gave for maximum matching (the dual of the LP for vertex cover) has a *half integral* (all variables receive values from $\{0, \frac{1}{2}, 1\}$) optimal solution. Prove that the same holds for the LP for vertex cover. Are all vertices of the polytope defined by the vertex cover LP half integral?

2. Prove that the vertex-arc incidence matrix for directed graphs is totally unimodular.