The simplex algorithm

May 1 and May 8, 2002

The **simplex** algorithm was designed by G. Danzig in 1947. It works remarkably fast in practice. The algorithm is composed of two phases, which in essence are:

Phase 1: Find a vertex of the polyhedron.

Phase 2: Reduce the value of the objective function by walking from vertex to vertex along the edges of the polyhedron, until a minimum vertex is found.

We will show how this geometric interpretation of the algorithm can be carried out algebraically. We shall briefly discuss issues such as handling degeneracies, pivot selection rules, and efficient implementations of basic iterations. We shall also present an alternative geometric representation of the simplex algorithm (from which "simplex" and "pivot" derive their names) via column geometry.

The number of arithmetic operations performed per pivot operation is $O(mn)$. Though the number of pivot operations is potentially exponential, in practice it is small (typically $O(m)$).

Homework

1. A. Hoffman designed an example showing that cycling can occur in the simplex algorithm. Can the following most simple form of cycling occur at a degenerate vertex: first x_1 enters the basis and x_2 leaves it; then x_2 enters the basis and x_1 leaves it? If yes, provide an example where this happens. If no, explain why not.

2. Let B be the matrix associated with a basic feasible solution, and recall that $\bar{c}_j = c_j - c_B^T B^{-1} A_j$ is the reduced cost of variable x_j. Show that in the column geomtry representation (where we also have the convexity constraint $\sum x_j = 1$, and $z = c^T x$) the distance in the z direction between the point (A_j^T, c_j) representing x_j and the dual plane corresponding to the basis variables of B is exactly $\bar{c}_j$.