Applications of LP duality

June 12, 2002

We will use LP duality to prove the minimax theorem for two person zero sum games. (Historically, the minimax theorem was proved before LP duality.) We will comment on the relevance of this theorem to proving lower bounds for randomized algorithms.

We shall then use LP duality in a natural way to develop a polynomial time algorithm for vertex cover on bipartite graphs. (Note that we also developed such an algorithm last semester.)

Vertex cover (the complement of independent set) can be formulated as:
minimize $\sum_{i=1}^{n} v_i$ *subject to*:
$v_i + v_j \geq 1$, for each edge (v_i, v_j),
$v_i \in \{0, 1\}$, for each vertex v_i.
NP-hardness of vertex cover implies that integer programming is NP-hard.

We relax the integer program to a linear program by replacing the $v_i \in \{0, 1\}$ constraints by nonnegativity constraints $v_i \geq 0$. The gap between the value of the optimal solution for integer program and its relaxed fractional program is known as the *integrality gap*. For the case of vertex cover, it is a multiplicative factor of at most 2.

By studying the dual of this linear program we will derive a polynomial time algorithm for vertex cover in bipartite graphs. In particular we will show that if all coefficients of a linear program are integer and the constraint matrix A is *totally unimodular*, then the linear program (if feasible) has an integer optimal solution.

Homework

We saw in class that the dual LP that we gave for maximum matching has a *half integral* (all variables receive values from $\{0, \frac{1}{2}, 1\}$) optimal solution. Show that the same holds for the LP for vertex cover. (One possible approach is to use duality.)

For extra credit: are all vertices of the polytope defined by the vertex cover LP half integral? Prove, or show a counter example.