Lecture 1

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1 The linear algebra of linear programs

Many optimization problems can be formulated as linear programs. The main features of a linear program are the following:

- Variables are *real numbers*. That is, they are continuous rather than discrete.
- The objective function is a linear function of the variables. (Each variable effects the object function linearly, at a slope independent of the values of the other variables.)
- Constraints on the variables are linear.

One of the first recorded examples of a linear program is the so called *diet* problem. Assume that a persons diet is composed entirely of n foods. Assume that there are m nutrients, and the person is required to consume at least b_i units of nutrient i (for $1 \leq i \leq m$). Let a_{ij} denote the amount of nutrient i present in one unit of food j. Let c_i denote the cost of one unit of food item i. The problem is to design a diet of minimal cost that supplies at least the required amout of nutrients.

Using matrix and vector representation, we have the following linear program:

minimize $c^T x$
subject to
$Ax \geq b$
$x \geq 0$

Let us remark that one could add constraints limiting from above the amount of nutritions consumed (e.g., not too much fat in the diet).

(Modeling the diet problem as a linear program includes many hidden assumptions such as that the cost is linear, that there are no interdependencies in the nutritional values of various foods, that any fractional amount of food can be bought, etc. As these assumptions are only approximations of the true state of affairs, an optimal solution to the linear program can only be viewed as a guideline for selecting a desireable solution, and not as the ultimate solution.)

A solution satisfying all constraints is *feasible*. A feasible solution that also optimizes the objective function is *optimal*.

1.1 Fourier-Motzkin elimination

In the coming lectures we will see approaches for solving linear programs. But first, let us sketch an inefficient approach.

Consider first the case of finding a feasible solution. If all constraints were equalities rather than inequalities, we could use Gaussian elimination (and find a feasible solution in polynomial time). For inequalities we may perform a similar (but less efficient) procedure known as Fourier-Motzkin elimination. To eliminate a variable x_i, we bring all inequalities in which it participates into one of two forms: $x_i \leq \alpha_j$ and $x_i \geq \beta_j$, where for every $1 \leq j \leq m$, α_j and β_j are linear functions of the rest of the variables. We can now eliminate x_i, and replace all constraints that included it by the new constraints $\beta_j \leq \alpha_k$ (for $1 \leq j \leq m$, $1 \leq k \leq m$). Repeating this procedure for the other variables gives eventually a set of inequlities in one variable, that can easily be solved. Note that each variable elimination at most squares the number of constraints. Hence the complexity of the above procedure is $O(m^{2^n})$ in the worst case. (Note however that if we can in each elimination step choose a variable that appears in at most 4 inequalities, then the number of inequalities does not grow, and the procedure becomes polynomial.)

Given the ability to find feasible solutions, we can in principle also find near optimal solutions. This can be done by replacing the objective function by a constraint $c^T x \leq t$, and searching for feasible solution. By doing binary search on t (assuming that we have reasonable estimates for lower and upper bounds of t), we can find a solution arbitrarily close to the true optimum. For the specific case of using Fourier-Motzkin elimination, we can treat t as a variable and leave it last in the elimination order. This allows us to find the exact optimum.

We shall now develop some theory that will allow us to design more efficient algorithms for solving linear programs.

1.2 Forms

The linear program for the diet problem is in *canonical form*. In a linear program in *general form*, the constraints are linear but may involve inequalities of both types ($\leq$ and $\geq$), as well as equalities ($=$). Variables may be required to be nonnegative ≥ 0, or else be unconstrained. Another useful form of a linear program is the *standard form*:

minimize $c^T x$
subject to
$Ax = b$
$x \geq 0$

All forms are equivalent in terms of their expressive power, and it is simple to transform a linear program in general form to standard form and to canonical form.

To replace an unconstrained variable x_i by nonnegative variables, introduce to auxiliary variables x_i^+ and x_i^-, replace every occurance of x_i by $x_i^+ - x_i^-$, and add the two nonnegativity constraints $x_i^+ \geq 0$ and $x_i^- \geq 0$.

To change a $\leq$ inequality $A_i^T x \leq b_i$ to a $\geq$ inequality, negate it, obtaining $-A_i^T x \geq -b_i$.

To change an equality $A_i^T x = b_i$ to inequalities, replace it by $A_i^T x \leq b_i$ and $A_i^T x \geq b_i$.

To change an inequality $A_i^T x \leq b_i$ to an equality, introduce a *slack variable* y_i, the inequality $A_i^T x + y_i = b_i$, and the nonnegativity constraint $y_i \geq 0$.

Note that the above transformations change a general linear program with n variables and m constraints to a linear program with $O(n+m)$ variables and constraints.

For linear programs in standard form, it is convenient to assume that the constraints (rows of the matrix A) are linearly independent. If the rows are not linearly independent, then it suffices to consider rows of A that constitute a basis for the row space (a maximal linearly independent set of row vectors). Either every solution that satisfies the constraints that correspond to the basis satisfies all constraints, or the LP is infeasible.

1.3 Basic feasible solutions

Consider an LP in standard form, with linearly independent constraints. Let B be a submatrix of A containing exactly m linearly independent columns. This is a *basis* of the column space of A. Let x_B be the set of basic variables corresponding to the columns of B. If $B^{-1}b \geq 0$, then the following is a basic feasible solution: the basic variables are set to $B^{-1}b$, and the nonbasic variables are set to 0. Clearly this solution is feasible. Note that it satisfies n linearly independent constraints with equality: the m constrains of $Ax = b$, and $n - m$ of the nonnegativity constraints. The other (nonnegativity) constraints are also satisfied, though not necessarily with equality.

Each basis gives at most one basic feasible solution. (It gives none if the condition $B^{-1}b \geq 0$ fails to hold.) Two different bases may give the same basic feasible solution, in which case the basic feasible solution is degenerate (more than $n-m$ variables are set to 0).

Basic feasible solutions play in important role in linear programs.

Lemma 1 *For every basic feasible solution, there is a choice of objective function $c^t x$ for which the bfs is the unique optimal solution.*

Proof: Consider an objective function for which $c_i = 0$ if x_i is positive in the bfs, and $c_i = 1$ otherwise. The objective function is nonnegative for every feasible solution (because of the nonnegativity constraints), and 0 for the particular bfs. As there is no other feasible solution with the same (or larger by containment) set of of 0 variables (otherwise $B^{-1}b$ would have more than one solution), the bfs is the unique minimum. $\square$

Lemma 2 *Every LP in standard form is either infeasible, or the optimal value is unbounded, or it has a basic feasible solution that is optimal.*

Proof: As the constraints of a linear program define a closed set (there are no strict inequalities), there is an optimal value and it is attained by some feasible solution (if the LP is feasible and the optimal value is unbounded). Let x be an optimal solution to the LP with the maximum number of 0 variables. Let x^+ be the set of positive variables in x. If the columns in A corresponding to x^+ are linearly independent, then we can complete them to a basis giving x as a bfs.

Assume that the columns in A corresponding to x^+ are not linearly independent. Denote them by $A^1, \dots, A^t$ and the positive variables by $z_1, \dots, z_t$. Then we have $\sum_{i=1}^t A^i z_i = b$. There is also a solution w to $\sum_{i=1}^t A^i x_i = 0$. Necessarily, $c^t w = 0$, because otherwise we could perturb x by a small multiple of w in the direction where $c^t w \leq 0$, and obtain a new feasible solution with smaller value, contradicting the optimality of x. Changing x by a multiple of w until the first time a new nonnegativity constraint becomes tight (this can be enforced, because w has at least one nonzero variables, and is zero whenever x is) gives a solution with more 0 variables and not higher value. $\square$

Remark: The above proof technique also shows that every LP in standard form that has a feasible solution and has no infinite line must also have a basic feasible solution. (In the above proof, change x by a multiple of w in the direction where this line in bounded, until the first time a new nonnegativity constraint becomes tight.)

Lemma 2 implies that in order to solve an LP optimaly, it suffices to consider only basic feasible solutions. As there are at most $\binom{n}{m}$ basic feasible solutions, we can solve LPs optimaly in this time.

Recall Cramer's rule for solving $Bx = b$, where B is an invertible order n matrix. The solution is

$$x_j = \frac{\det B^j}{\det B}$$

for $1 \leq j \leq n$, where here B^j is the matrix B with column j replaced by b. If each entry in B and b is an integer with absolute value at most M, then each x_j is a rational number with numerator and denominator bounded by at most $M^n n!$. This can be used to show that the length of numbers involved in a basic feasible solution are polynomially related to the input size. (Moreover, it can be shown that when a system of linear equations is solved by Guassian elimination, the length of intermediate numbers produced by the algorithm is also polynomially related to the input size.)