Approximation algorithms using semidefinite programming

QUESTIONS. Possibly, the number of questions is too large to be solved within the time allocated. Hints to possible solutions are given in page 3.

We considered the Linear Program for Max cut. The *odd cycle constraints* say that for every odd cycle of length g, the sum of values of edges along the cycle is at most $g-1$ (because every cut cuts at most $g-1$ of these edges). For this linear program to be solvable in polynomial time, we want a polynomial time procedure to find violated constraints.

Question 1: For a given graph G and a given assignment of values in the range between 0 and 1 for the edges, show that there is a polynomial time algorithm that finds whether there is an odd cycle of length g in which the sum of values along its edges exceeds $g-1$.

The LP with odd cycle constraints characterizes the value of max-cut for every planar graph. Observe that a feasible solution to the LP is to give every edge value $2/3$. It follows that every planar graph has a cut with at least $2|E|/3$ edges.

Question 2: Give an alternative proof that in every planar graph the optimal solution to max cut is at least $2|E|/3$.

Question 3: Show that every graph of maximum degree d has a cut with at least $(\frac{1}{2}+\frac{1}{2(d+1)})|E|$ edges.

A feasible solution to the minimization dual of the SDP for max-cut is to add $s_1,\dots,s_n$ along the diagonal of the adjacency matrix of the graph such that it becomes positive semidefinite (no negative eigenvalues). A value of s_i on entry A_{ii} on the diagonal can intuitively be thought of as placing $s_i/2$ self loops on vertex i.

Question 4: Show that for the graph representing this feasible solution, every cut cuts at most half the edges of every vertex induced subgraph (the value $s_i/2$ of self-loops contributes to the part not being cut).

The triangle constraints $x_ix_j+x_jx_k+x_kx_i \geq -1$ and $x_ix_j-x_jx_k-x_kx_i \geq -1$ are very useful in SDP formulations of many problems.

Question 5: Show that for max-cut the triangle constraints imply all odd cycle constraints. Namely, the edges of an odd cycle of length g can contribute at most $g-1$ to the value of the SDP.

Geometrically, the triangle constraints imply that as points in space, x_i, x_j, x_k are never allowed to form a triangle with an obtuse angle, and the same holds for $x_i, x_j, -x_k$. This is so as they are equivalent to the constraint that the square of distances between every two points form a metric (satisfy the triangle inequality), combined with Pythagoras' theorem. (The distances themselves of course form a metric even without the triangle constraints.)

Question 6: Show that if three unit vectors in the plane satisfy the triangle constraints (a 2-dimensional solution), then necessarily at least two of them lie on the same line (either in identical or opposite directions).

The results of question 6 can be used to show that whenever an optimal solution to the SDP for max cut with triangle constraints is in two dimensions, the value of the sdp is exactly equal to the max-cut. (For one dimension this is true even without triangle constraints.)

Recall that for 2SAT, we introduced a unit vector x_0 (which is interpreted as the $+1$ direction), and a clause $(x_i \vee x_j)$ contributes

$$\frac{4 - (x_0 - x_i)(x_0 - x_j)}{4} = \frac{3 + x_0 x_i + x_0 x_j - x_i x_j}{4}$$

to the objective function of the SDP. (Here $x_i = 1$ corresponds to a variable set to true and $x_i = -1$ corresponds to a variable set to false.)

Question 7: Show that without triangle constraints there are feasible solutions to the SDP in which a clause contributes more than 1 to the objective function, whereas this cannot happen with triangle constraints.

The problem of min bipartization (remove the smallest possible number of edges so that the graph becomes bipartite) is the complement of max-cut. Nevertheless, the SDP does not provide a constant factor approximation for it.

Question 8: Show that asymptotically, as ϵ tends to 0, if $\frac{1-x_i x_j}{2} = 1-\epsilon$, then the angle between unit vectors x_i and x_j is $(1-\Omega(\sqrt{\epsilon}))\pi$.

Proper/nonproper max k-sat is the problem of finding an assignment that maximizes the number of clauses satisfied, where each clause contains exactly/at most k literals (belonging to distinct valiables). We saw that SDP can be used to give an 0.94 approximation ratio of max 2sat (proper or not).

Question 9: Show that in every proper 3CNF formula at least 7/8 of the clauses are satisfiable.

Hence it is easy to approximate proper max 3sat within a ratio of 7/8. Using SDP one can approximate nonproper max 3sat within a ratio of 7/8. The difficulty in approximating the nonproper version is that the approximation algorithms for max 2sat and for proper max 3sat are of a different nature, and it is not easy to see that they mix well with each other. The proof uses computer assisted analysis.

HINTS.

1. For an edge e_i of value $0 \leq w_i \leq 1$, let its length be $l_i = 1 - w_i$. Construct 2 copies V_1 and V_2 of the vertex set of the graph, connect them by directed edges, and find a shortest path between the copies of vertex v in V_1 and V_2.

2. The famous 4-color theorem says that every planar graph is 4-colorable.

3. Again, you may use coloring.

4. Use Raleigh quotients.

5. Represent the edges of an odd cycle as a sum of triangles, using both types of triangle constraints to induce cancelations.

8. The inner product of two unit vectors is the cosine of the angle between them. An alternative view: the objective function can be rewritten as $(x_i - x_j)^2/4$.

9. Consider a random assignment.

RESEARCH QUESTIONS. If you solve any of the open questions below, please let me know.

1. Give a rounding technique for the dual minimization SDP for max cut (referred to in question 4) with approximation ratio $\alpha \simeq 0.878$.
2. Determine the worst integrality ratio for the SDP for max-cut with triangle constraints. (It is know to be between $\simeq 0.878$ and $\simeq 0.891$.)
3. Currently, the best known approximation ratios for max bisection are far behind those for max cut. Either significantly improve these ratios, or give evidence that indeed max bisection is strictly harder to approximate than max-cut.
4. Either give an algorithm that approximates min-bipartization within a constant ratio, or show that for every constant ρ, achieving an approximation ratio of ρ is NP-hard.