Games For Arbitrarily Fat Rats

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Abstract

We study rational Beatty sequences that partition the natural numbers in any given finite dimension, introduced by Fraenkel. We prove that there is a 2-player vector subtraction game, as introduced by Golomb, that admits any such given sequence, together with the **0**-vector, as its unique set of P-positions.

1 Introduction

We set out to blend recent ideas in combinatorial number theory (CNT) with a modern trend in combinatorial game theory (CGT).

Regarding CNT, *Beatty sequences* are normally associated with irrational *moduli* α, β. Recent studies deal with rational moduli α, β. Clearly if $a/b \neq g/h$ are rational, then the sequences $\{\lfloor na/b \rfloor\}$ and $\{\lfloor ng/h \rfloor\}$ cannot be complementary, since $kbg \times a/b = kha \times g/h = kag$ for all $k \geq 1$. Also the former sequence is missing the integers $ka - 1$ and the latter $kg - 1$, so both are missing the integers $kag - 1$ for all $k \geq 1$. However, complementarity can be maintained for the *nonhomogeneous* case: In [12], [29], necessary and sufficient conditions on α, γ, β, δ are given so that the sequences $\{\lfloor n\alpha + \gamma \rfloor\}$ and $\{\lfloor n\beta + \delta \rfloor\}$ are complementary – for both irrational moduli and rational moduli. We are not aware of any previous work in this

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direction, except that in Bang [3] necessary and sufficient conditions are given for $\{\lfloor n\alpha \rfloor\} \supseteq \{\lfloor n\beta \rfloor\}$ to hold, both for the case α, β irrational and the case α, β rational. Results of this sort also appear in Niven [28], for the homogeneous case only. In Skolem [32] and Skolem [33] the homogeneous and nonhomogeneous cases are studied, but only for α and β irrational.

These investigations spawned the following conjecture [13], Erdös and Graham [11]: If the system $\cup_{i=1}^{m}\{\lfloor n\alpha_i + \gamma_i \rfloor\}$, $n = 1, 2, \ldots$ splits the positive integers with $m \geq 3$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_m$, then

$$\alpha_i = (2^m - 1)/2^{m-i},\ i = 1, \ldots, m. \tag{1}$$

It is well-known that if all the α_i are integers with $m \geq 2$ and $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m$, then $\alpha_{m-1} = \alpha_m$. A proof using complex numbers and roots of unity was given by Mirsky, Newman, Davenport and Rado – see Erdös [10]. A first elementary proof was given independently in [5] and by Simpson [30]. Graham [19] showed that if all the m moduli are irrational and $m \geq 3$, then two moduli are equal. Thus distinct integer moduli or irrational moduli cannot exist for $m \geq 2$ or $m \geq 3$ respectively in a splitting system.

The conjecture was proved for $m = 3$ by Morikawa [26], $m = 4$ by Altman et. al [2], for all $3 \leq m \leq 6$ by Tijdeman [35] and for $m = 7$ by Barát and Varjú [4] and was generalized by Graham and O'Bryant [20]. Other partial results were given by Morikawa [27], Simpson [31]. Many others have contributed partial results – see Tijdeman [34] for a detailed history. The conjecture has some applications in job scheduling and related industrial engineering areas, see e.g., Altman et. al [2], Brauner and Jost [6]. However, the conjecture itself has not been settled. So this is a problem that has been solved for the integers, has been solved for the irrationals, and is wide open for the rationals!

The conjecture induced the rat game and its associates the mouse game and the fat rat game [16] (rat – rational). The rat game, mouse game, fat rat game are played on 3 piles, 2 piles and 4 piles of tokens respectively, whose P-positions are the cases $m = 3, 2, 4$ of the conjecture respectively, together with $\mathbf{0}$. Thus, for the rat game, the P-positions are $\{(\lfloor 7n/4 \rfloor, \lfloor 7n/2 \rfloor, 7n - 3),\ n = 1, 2, \ldots\} \cup \{\mathbf{0}\}$. For the rat and mouse games we also gave game rules, but for the fat rat game, no game rules were found.

This brings us to a modern trend in CGT. A typical interest in CGT is, given a finite rule set describing a game, find its P-positions, or also, when possible, its Sprague-Grundy function. A modern trend is to reverse this process: given a subsequence R of nonnegative vectors, is there a game whose set of P-positions is precisely R? Any game for which some game-move cannot be made from all game-positions because it would be a move connecting two P-positions, is a *variant* game. Duchêne and Rigo [9] conjectured that if R is the set of numbers produced by a pair of complementary homogeneous Beatty sequences (with irrational moduli), then there is an *invariant* game whose set of P-positions is R, together with $(0, 0)$. Larsson et. al proved a generalization thereof [23]. Informally, a game is invariant if every move can be done from every position, provided only that the result is a game position.

This opens up the possibility of proving *existence* of games, though their game rules are not necessarily known; nor even if there exist finite[1] game rules. The situation is analogous to Erdös' probabilistic method [1]: to prove that some system with desired properties exists, a probability space of systems is defined, proving that those properties hold in this space with positive probability.

2 Preliminaries

Throughout, both game-positions and game-moves are m-dimensional non-negative integer vectors. So the terms "positions", "moves", "vectors" may be used interchangeably below.

In an attempt to formalize the concept of reasonable, interesting, appealing games, Duchêne and Rigo [9] defined the notion of invariant games, restricted to vector subtraction games.

We write $\boldsymbol{x} \preceq \boldsymbol{y}$ if $x_i \leq y_i$ for all $1 \leq i \leq m$; and $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $x_i < y_i$ for some i. The game G is *invariant* if for all positions $\boldsymbol{p}$ and $\boldsymbol{q}$ and any move $\boldsymbol{x}$ such that $\boldsymbol{x} \preceq \boldsymbol{p}$ and $\boldsymbol{x} \preceq \boldsymbol{q}$, the move $\boldsymbol{p} \to \boldsymbol{p} - \boldsymbol{x}$ is permissible if and only if the move $\boldsymbol{q} \to \boldsymbol{q} - \boldsymbol{x}$ is permissible. Otherwise G is *variant*. Invariant games are the games considered by Golomb [18]. Any move in an invariant (variant) game is called invariant (variant). Duchêne and Rigo readily admitted that there are appealing variant games when "the dependence of the game rules to the actual positions is restricted to some simple logical formula".

Notation 1. (i) For any m-dimensional vector $\boldsymbol{a} = (a(1), \ldots, a(m))$, we assume $a(1) \leq \cdots \leq a(m)$.

(ii) The set of all P-positions of a game is denoted by $\mathcal{P}$.

In a variant game, the vector $\boldsymbol{x} - \boldsymbol{y}$ may be permitted as a move 'somewhere else', even if $\boldsymbol{x}$ and $\boldsymbol{y}$ are both in $\mathcal{P}$. Such a move is not permitted

[1] If you can write down in a finite amount of time.

in an invariant game. This simple observation motivates the second item of Definition 1 below.

Definition 1. (i) An integer vector $\boldsymbol{s}$ is *feasible* in a subtraction game, if $s(i) \geq 0$ for all i and $s(i) > 0$ for some i.

(ii) A feasible vector $\boldsymbol{s}$ is *applicable* to a vector $\boldsymbol{v}$ if $\boldsymbol{v} - \boldsymbol{s} \succeq \mathbf{0}$.

(iii) Let R be candidate set of P-positions in a vector subtraction game. Any vector that connects two vectors in R, that is, any vector $\boldsymbol{r} - \boldsymbol{r}'$ with $\boldsymbol{r}, \boldsymbol{r}' \in R$, is *inadmissible*. Any other vector is *R-admissible*.

(iv) If a vector is both feasible and admissible, we call it a *move*.

(v) A move $\boldsymbol{s}$ is *befitting* for a vector $\boldsymbol{v}$ if $\boldsymbol{v} - \boldsymbol{s} \succeq \mathbf{0}$.

Example 1. $m = 2$. $(-1, 4)$ is not feasible, but $(3, 5)$ is feasible. Further, $(3, 5)$ is applicable to $(3, 7)$ because $(3, 7) - (3, 5) = (0, 2) \succeq \mathbf{0}$. But $(3, 5)$ is not applicable to $(2, 7)$.

Notice that in every invariant game, every move is, in particular, admissible (for $R = \mathcal{P}$).

We will prove that for every $m \geq 2$ there exists an invariant two-player vector subtraction game, with admissible moves only, that admits any given sequence based on reals of the form (1), as specified below, together with the $\mathbf{0}$-vector, as its unique set of P-positions.

For the purpose of the present work, we use the normal-play convention: a player who cannot subtract, because at least one of the coordinates would become negative, loses.

Since we disallow inadmissible vectors as moves, Definition 1 (ii) provides the *independence* property of a sequence of P-positions of a subtraction game: there is no move within R. Thus, for R to be a complete sequence of P-positions, it suffices to demonstrate, that from any vector not in R, there is an admissible move to a position in R.

3 Rat games

Notation 2. Let $\alpha_{m,k} = (2^m - 1)/2^{m-k}$, $\beta_k = -2^{k-1} + 1, \quad k = 1, \ldots, m,$

Let us now define our class of subsequences. Given a finite dimension $m \geq 2$, the standard form for our m-dimensional sequence is:

$$\left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor - 2^{k-1} + 1, \quad k = 1, \ldots, m, \quad n = 1, 2, \ldots,$$

that is, for each $m \geq 2$, $n \geq 1$, the *rats* $\boldsymbol{r}_n$ are,

$$\boldsymbol{r}_n = \left(\left\lfloor \frac{2^m - 1}{2^{m-1}} n \right\rfloor, \left\lfloor \frac{2^m - 1}{2^{m-2}} n \right\rfloor - 1, \ldots, (2^m - 1)n - 2^{m-1} + 1 \right), \qquad (2)$$

$n = 1, 2, 3, \ldots$, together with $\boldsymbol{r}_0 := \mathbf{0}$. We write $R = \{\boldsymbol{r}_n\}_{n=0}^{\infty}$. The components of $\boldsymbol{r}_n$ are $r_n^k := \lfloor (2^m - 1)n/2^{m-k} \rfloor - 2^{k-1} + 1$, $k = 1, \ldots, m$. It is known [13] that for every $m \geq 2$, the set of rats $\{\boldsymbol{r}_n\}_{n=1}^{\infty}$ partitions the positive integers. (The case $m = 1$ is trivial and is therefore excluded.)

Lemma 1. (i) $\boldsymbol{r}_1 = (1, 2, 4, \ldots, 2^{m-1})$.

(ii) *For any inadmissible (vector)* $\boldsymbol{s}$ *not connecting to* $\mathbf{0}$, *there exist integers* $1 \leq n_1 < n_2$ *such that* $s(m, i) = \lfloor (n_2 \alpha_{m,i}) \rfloor$ - $\lfloor n_1 \alpha_{m,i} \rfloor$, $i = 1, \ldots, m$. *In particular,* $s(m, m) = (n_2 - n_1)(2^m - 1) = (n_2 - n_1)\alpha_{m,m}$.

(iii) *For* $n \geq 1$, $r_n(m) = 2^{m-1} + (n - 1)(2^m - 1)$.

Proof. (i) $\lfloor (2^m - 1)/2^{m-i} \rfloor - 2^{i-1} + 1 = (2^i - 1) - 2^{i-1} + 1 = 2^{i-1}$ for $1 \leq i \leq m$.

(ii) Any move within $R \setminus \{\mathbf{0}\}$ must satisfy, for the m-th component,

$$(2^m - 1)n_2 - 2^{m-1} + 1 - (2^m - 1)n_1 + 2^{m-1} - 1 = (n_2 - n_1)(2^m - 1)$$

for some $n_2 > n_1 > 0$. The same argument holds for the other components $s(i)$.

(iii) For $n = 1$ it follows from (i). If true for n, then for $n + 1$ it follows from (ii). ■

The following lemma is one of our main results.

Lemma 2. *Given any dimension* $m \geq 2$, *there is a feasible and admissible move from every* m*-dimensional position* $\boldsymbol{x}$ *not in* $\mathcal{R}$, *to a position in* $\mathcal{R}$.

Proof. Let $\boldsymbol{x}$ be a nonnegative m-dimensional vector, $m \geq 2$, $\boldsymbol{x} \notin \mathcal{R}$. We have to find a move from $\boldsymbol{x}$ to a position in $\mathcal{R}$. There are two possibilities, either $\boldsymbol{x}$ is admissible (regarded as a move) or $\boldsymbol{x}$ is inadmissible.

Observe that because $\boldsymbol{x} \neq \mathbf{0}$ is a position, it is trivially feasible.

First suppose that the position $\boldsymbol{x}$ is admissible. Then $\boldsymbol{x} - \boldsymbol{x} = \mathbf{0}$, and we are done with this case, since $\mathbf{0} \in \mathcal{R}$ (normal play).

Suppose next that $\boldsymbol{x} \notin \mathcal{R}$ is inadmissible. We have to find a position $\boldsymbol{r} \in R$, such that $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{s}$, where $\boldsymbol{s}$ is a feasible and admissible move.

The position $\boldsymbol{r}$ we seek is not $\mathbf{0}$, since $\boldsymbol{r} = \mathbf{0} \implies \boldsymbol{x} = \boldsymbol{s}$, so the move $\boldsymbol{s}$ would be inadmissible. Since $\boldsymbol{x} \notin R$ is inadmissible, there exist vectors $\boldsymbol{r}$, $\boldsymbol{r}' \in R$ such that $\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{r}'$. If $\boldsymbol{r}' = \mathbf{0}$, then $\boldsymbol{x} = \boldsymbol{r} \in R$, contradicting $\boldsymbol{x} \notin R$. Thus $\boldsymbol{r}' \neq \mathbf{0}$. Therefore Lemma 1(ii) implies that there exist integers $1 \leq n_1 < n_2$ such that $x(i) = \lfloor n_2\alpha_i \rfloor - \lfloor n_1\alpha_i \rfloor$, $i = 1, \ldots, m$, so

$$x(m) = (n_2 - n_1)(2^m - 1).$$

Observe that $x(m) = r_n(m)$ for no $n \geq 1$. For suppose $x(m) = r_n(m)$. Then Lemma 1(iii) implies $(n_2 - n_1)(2^m - 1) = 2^{m-1} + (n - 1)(2^m - 1)$, so $2^{m-1} = (n_2 - n_1 - n + 1)(2^m - 1)$. For $m > 1$, this is impossible, since the right-hand-side has an odd factor (namely $2^m - 1$), and because $n_2 - n_1 - n + 1 \neq 0$. This odd factor is absent from the left-hand side.

Further, notice that $x(m) = (n_2 - n_1)(2^m - 1) \geq 2^m - 1 > 2^{m-1} = r_1(m)$ for $m \geq 2$. Since $x(m) = r_n(m)$ for no $n \geq 1$, there exists $n_0 \geq 1$ such that $r_{n_0}(m) < x(m) < r_{n_0+1}(m)$. Thus by Lemma 1(iii),

$$2^{m-1} + (n_0 - 1)(2^m - 1) < (n_2 - n_1)(2^m - 1) < 2^{m-1} + n_0(2^m - 1).$$

Dividing by $2^m - 1$ and rearranging terms, we get:

$$-1 < \frac{2^{m-1}}{2^m - 1} - 1 < n_2 - n_1 - n_0 < \frac{2^{m-1}}{2^m - 1} < 1.$$

Since the three n_i are integers, we conclude

$$n_2 - n_1 = n_0.$$

We now show that the move $\boldsymbol{x} \rightarrow \boldsymbol{r}_{n_0}$, namely $x(i) \rightarrow r_{n_0}(i)$, $i = 1, \ldots, m$ is both feasible and admissible. We begin with feasability.

The preceding arguments show that the coordinate move $x(i) \rightarrow r_{n_0}(i)$ can be written in the form $x(i) = \lfloor n_2\alpha_i \rfloor - \lfloor n_1\alpha_i \rfloor \rightarrow \lfloor n_0\alpha_i \rfloor + \beta_i = r_{n_0}(i)$. The definition of the floor function implies that

$$\lfloor n_2\alpha_i \rfloor - \lfloor n_1\alpha_i \rfloor \geq \lfloor (n_2 - n_1)\alpha_i \rfloor = \lfloor n_0\alpha_i \rfloor \geq \lfloor n_0\alpha_i \rfloor + \beta_i,$$

where the last inequality is strict for all $i > 1$, since $\beta_i < 0$ for $i > 1$. Thus the move is feasible.

For admissibility, notice that

$$x(m) - r_{n_0}(m) = (n_2 - n_1)(2^m - 1) - 2^{m-1} - (n_0 - 1)(2^m - 1) = 2^{m-1} - 1.$$

By Lemma 1(ii), $s(m) = n_0(2^m - 1) \geq 2^m - 1 > 2^{m-1} - 1$. Thus the move is admissible. ∎

We conclude:

Theorem 1. *For every $m \geq 2$, there is an invariant vector subtraction game. The rule set is to subtract any feasible admissible m-tuple, namely, $\boldsymbol{s} \neq \boldsymbol{r}_i - \boldsymbol{r}_j$, $\boldsymbol{r}_i, \boldsymbol{r}_j \in R$, for any i, j. If $\boldsymbol{s} = \boldsymbol{r}_i - \boldsymbol{r}_j$ for some $\boldsymbol{r}_i, \boldsymbol{r}_j \in R$, then $\boldsymbol{s}$ is not a move.*

Proof. By Definition 1, there is no admissible move from a position in R to another position in R. By Lemma 2, there is a feasible and admissible move from any position not in R to some position in R. Thus R satisfies the conditions for a unique set of P-positions. The game is invariant since the moves do not depend on the positions. ■

Remark 1. (i) The rule set of every game is a subset of all of its admissible vectors, where the subset is determined by the game's rule set. In the present case, where there is only the game's P-positions, the rule set is to use *all* admissible vectors as moves.

(ii) We do not know whether there exist invariant rule sets for our rat games that do not allude to their sets of P-positions. In [16] such game rules were given for the cases $m = 2$ and $m = 3$, but they were variant.

(iii) To make Theorem 1 usable, we need an efficient algorithm to decide whether any given feasible m-tuple is admissible or not. This will be taken up in section 5, where we will also compute the number of inadmissible moves. In the next section we prepare the ground by studying basic properties of the rats.

4 Anatomy of the rats

The P-positions of the form (2) are not too convenient to work with, mainly because of the floor function. There is an equivalent matrix representation that captures the P-positions (2). We begin with an example, $m = 4$, which was dubbed fat rat in [16].

Example 2. The standard form of the P-positions for $m = 4$, without $\mathbf{0}$, $n \geq 1$,

$$\boldsymbol{r}_n = \boldsymbol{r}_n^4 = \left(\left\lfloor \frac{15}{8} n \right\rfloor, \left\lfloor \frac{15}{4} n \right\rfloor - 1, \left\lfloor \frac{15}{2} n \right\rfloor - 3, 15n - 7 \right). \qquad (3)$$

The first 11 rows of the rat matrix, $t = \lfloor (n-1)/2^{m-1} \rfloor$, $m = 4$, $n \geq 1$,

$$\mathcal{R}_4 = \begin{pmatrix} n & r_n^1 & r_n^2 & r_n^3 & r_n^4 \\ 1 & 15t+1 & 30t+2 & 60t+4 & 120t+8 \\ 2 & 15t+3 & 30t+6 & 60t+12 & 120t+23 \\ 3 & 15t+5 & 30t+10 & 60t+19 & 120t+38 \\ 4 & 15t+7 & 30t+14 & 60t+27 & 120t+53 \\ 5 & 15t+9 & 30t+17 & 60t+34 & 120t+68 \\ 6 & 15t+11 & 30t+21 & 60t+42 & 120t+83 \\ 7 & 15t+13 & 30t+25 & 60t+49 & 120t+98 \\ 8 & 15t+15 & 30t+29 & 60t+57 & 120t+113 \\ 9 & 15t+1 & 30t+2 & 60t+4 & 120t+8 \\ 10 & 15t+3 & 30t+6 & 60t+12 & 120t+23 \\ 11 & 15t+5 & 30t+10 & 60t+19 & 120t+38 \end{pmatrix}.$$

The reader is encouraged to check that the values of $\boldsymbol{r}_n$, as n ranges from 1 to 11, are identical to the 11 lines of the matrix. For example, for $n = 6$, the value of (3) is $(11, 21, 42, 83)$, the same as the line $n = 6$, $t = \lfloor 6/8 \rfloor = 0$ of $\mathcal{R}_4$. For $n = 9$, (3 yields $(16, 32, 64, 128)$, same as line 9 of $\mathcal{R}_4$ with $t = \lfloor 9/8 \rfloor = 1$. Also notice the periodicity mod 15 in the first column, after the first 8 rows, and analogous periodicities in the other 3 columns (Fact 1 below).

More generally, for any $m \geq 2$, define the $2^{m-1} \times m$ matrix $\mathcal{R}_m = (r_{i,j})$, whose elements are defined by:

$$r_{i,j} = 2^{j-1}(2^m - 1)n + 2^{j-1} + (i-1)2^j - \lceil i/2^{m-j} \rceil + 1, \tag{4}$$

$i = 1, \ldots, 2^{m-1}$, $j = 1, \ldots, m$, $n = 1, 2, \ldots$.

Notice, for example, that for $m = 4$, $r_{i,j}$ produces precisely the values of the matrix of Example 2.

Lemma 3. *For every $m \geq 2$, the vectors $\boldsymbol{r}_n$ of the form (2) and the matrix $\mathcal{R}_m$ produce identical outputs. Specifically, the 2^{m-1} vectors $\boldsymbol{r}_n$, as n ranges over $2^{m-1}k + 1$ to $2^{m-1}k + 2^{m-1}$, are identical to the 2^{m-1} rows of $\mathcal{R}_m$ for $n = k$ ($k \geq 0$).*

For proving this lemma, we begin by collecting a few facts about the rats $\boldsymbol{r}_n$.

Fact 1. Periodicity property. For every $1 \leq k \leq m$,

$$\left\lfloor \frac{2^m - 1}{2^{m-k}}(n + 2^{m-k}) \right\rfloor = \left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor + 2^m - 1,$$

so

$$\left\lfloor \frac{2^m-1}{2^{m-k}}(n+2^{m-k}) \right\rfloor \equiv \left\lfloor \frac{2^m-1}{2^{m-k}}n \right\rfloor \pmod{(2^m-1)}.$$

Thus, $\lfloor (2^m-1)n/2^{m-k} \rfloor$ is periodic mod 2^m-1 after 2^{m-k} consecutive values of n.

Fact 2. The structure of the $2^{m-1}-1$ row gaps. For $1 \le n < 2^{m-1}$ let,

$$\Delta_{n,k} := \left\lfloor \frac{2^m-1}{2^{m-k}}(n+1) \right\rfloor - \left\lfloor \frac{2^m-1}{2^{m-k}}n \right\rfloor.$$

For $k = m$, no floors are needed, and $\Delta_{n,m} = 2^m - 1$ for all $n \ge 1$. We may thus assume that $1 \le k < m$. For reals x, y, the floor function basic property implies $\lfloor x-y \rfloor \le \lfloor x \rfloor - \lfloor y \rfloor \le \lfloor x-y \rfloor + 1$. Also, $\lfloor (2^m-1)n/2^{m-k} \rfloor = 2^k n + \lfloor (-n)/2^{m-k} \rfloor$. Thus, $2^k - 1 \le \Delta_{n,k} \le 2^k$ for all $n \ge 1$.

We next determine for which values of n the gaps assume the value 2^k-1, and for which values 2^k is assumed.

The periodicity implies that $\Delta_{n+2^{m-k},k} = \Delta_{n,k}$. Hence it suffices to consider n in the integer interval $I := [1, 2^{m-k}]$. Suppose that for x values n in I the gap 2^k is assumed. Then the gap 2^k-1 is assumed for $2^{m-k}-x$ values of n. Thus, $2^k x + (2^k-1)(2^{m-k}-x) = 2^m - 1$. Solving gives $x = 2^{m-k}-1$. So only once in I is the gap $2^k - 1$ assumed. Now

$$\Delta_{2^{m-k},k} = \left\lfloor \frac{2^m-1}{2^{m-k}}(2^{m-k}+1) \right\rfloor - \left\lfloor \frac{2^m-1}{2^{m-k}}2^{m-k} \right\rfloor = 2^k - 1.$$

Thus the so-called *deficient* gap, of length 2^k-1, is assumed at the end of I. We have proved:

Lemma 4. *$\Delta_{n,k} = 2^k - 1$, the deficient gap, occurs only for $n \equiv 0 \pmod{2^{m-k}}$; $\Delta_{n,k} = 2^k$ for all other values of $n \ge 1$.*

This lemma gives us a convenient matrix form of the rats' P-positions.

Fact 3. We give two characterizations for the structure of the $m-1$ column gaps.

Lemma 5. *For $1 \le j < m$ and $1 \le n \le 2^{m-1}$, $r_n^{j+1} - r_n^j \in \{r_n^j, r_n^j - 1\}$. Moreover:*

(i) $r_n^{j+1} = 2r_n^j$ for $1 \le n \le 2^{m-j-1}$; $r_n^{j+1} = 2r_n^j - 1$ for $2^{m-j-1} < n \le 2^{m-j}$.

(ii) The binary representation $b(n-1)$ of $n-1$ indicates which of the two values is assumed: if $b(n-1)$ has a 0 in column j, then $r_n^{j+1} - r_n^j = r_n^j$; if it has a 1 in column j, then $r_n^{j+1} - r_n^j = r_n^j - 1$.

Sketch of proof for (i): From 4, the general term of matrix $\mathcal{R}_m$,

$$r_{i,j+1} - r_{i,j} = 2^{j-1} + (i-1)2^j - c_{i,j},$$

where $c_{i,j} = \lceil i/2^{m-j-1} \rceil - \lceil i/2^{m-j} \rceil$, and it remains only to analyze $c_{i,j}$.

Fact 4. For $N \geq 1$, let

$$\Delta_{n,k,N} := \left\lfloor \frac{2^m - 1}{2^{m-k}}(n+N) \right\rfloor - \left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor.$$

Clearly $\Delta_{n,k,1} = \Delta_{n,k}$.

The number D_k of deficient gaps in any stretch of length N depends on the location of the stretch. Without loss of generality (periodicity, Fact 1), we assume that the stretch begins at some $n \leq 2^{m-1}$. If the stretch begins at $n_0 \equiv 0 \pmod{2^{m-k}}$, then $D_k = \lceil N/2^{m-k} \rceil$, and this holds also for up to some critical height $H_k \leq n_0$. For $n < H_k$, $D_k = \lceil N/2^{m-k} \rceil - 1$.

Needs some justification

5 A constructive epilogue

Let $m \geq 2$ be fixed. Let $\boldsymbol{x} = (x_1, \ldots, x_m) \succeq \mathbf{0}$ with $x_1 \leq \cdots \leq x_m$ be a feasible vector. We seek an efficient algorithm to decide an optimal move from $\boldsymbol{x}$ for the next player – the one that plays from $\boldsymbol{x}$. We check, term by term, whether $\boldsymbol{x} = r_n$ (of the form (2)) for some $n \geq 1$. If so, $\boldsymbol{x}$ is a P-position, so Alice removes a single token from one of the piles. This is clearly an admissible move, and has the advantage of forcing Bob to compute his next move from a largest possible position.

So we may assume that $\boldsymbol{x} \notin \mathcal{P}$. Then $\boldsymbol{x} \in \mathcal{N}$. If $\boldsymbol{x}$ is admissible, then one option of Alice is to move $\boldsymbol{x} \to \mathbf{0}$, consistent with the rule set specified by Theorem 1, since $\boldsymbol{x}$ is admissible. Alice might choose an appropriate different P-option, if available, so as to prolong the agony of Bob, if she is so inclined. Notice that any such subtraction vector is necessarily admissible.

So assume that $\boldsymbol{x}$ is an inadmissible N-position. Then a subtraction $\boldsymbol{x} \to \mathbf{0}$ is prohibited, since $\boldsymbol{x} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{P}$, so $\boldsymbol{r}_2 \to \boldsymbol{r}_1$ is prohibited. For an invariant game, this excludes using the vector $\boldsymbol{r}_2 - \boldsymbol{r}_1$ as a move anywhere. In particular, $\boldsymbol{r}_2 - \boldsymbol{r}_1 \to \mathbf{0}$ is prohibited. Again for an invariant game, there must be an admissible subtraction vector leading from $\boldsymbol{x}$ to a P-position. Two problems have thus to be solved:

(i) Decide efficiently whether or not $\boldsymbol{x}$ is inadmissible.

(ii) If inadmissible, find efficiently a move to a P-option.

(i) We describe an algorithm for deciding this question, illustrating it by using Example 2. This algorithm is our second main result. We appeal to Facts 1, 2 and 4. For any integer interval $I = [a, b]$, we define $I + d = [a + d, b + d]$. Assume $\boldsymbol{x} = (9, 19, 37, 75)$.

Recall that $\boldsymbol{x}$ is inadmissible if $\boldsymbol{x} = \boldsymbol{r}_k - \boldsymbol{r}_j$ for some $\boldsymbol{r}_j, \boldsymbol{r}_k \in \mathcal{P}$. For $\boldsymbol{x}$ to be inadmissible, it is needed that $x_m = N(2^m - 1)$ for some $N \geq 1$. For our example, $N = 5$, so inadmissibility of $\boldsymbol{x}$ has not been ruled out. We now examine $x_1, \ldots, x_{m-1}$ in this order, testing for inadmissibility. We need $k - j = N$ for all components x_i of $\boldsymbol{x}$. Write $N = n_2 - n_1$ for arbitrary $1 \leq n_1 < n_2$.

For $k = 1$, is there $n_1 \geq 1$ such that $\Delta_{n_1,1,N} = x_1$? By Facts 2 and 4, the only candidates for x_1 such that $\boldsymbol{x}$ be inadmissible are $x_1 = 2N - D_1$, $D_1 \in \{\lceil N/2^{m-1} \rceil, \lceil N/2^{m-1} \rceil - 1\}$. For the example, $D_1 \in \{1, 0\}$, so $\Delta_{n_1,1,5} = 10$ if $D_1 = 0$; 9 if $D_1 = 1$. Notice that for $1 \leq n \leq 3$, $\Delta_{n,1,5} = 10$, but for $4 \leq n \leq 8$, $\Delta_{n,1,5} = 9$. We write $I_1 = [4, 8] + 8r_1$, $r_1 = 0, 1, 2, \ldots$, for the intervals I_1 of n-values accommodating $x_1 = 9$. Without loss of generality, $r_1 = 0$. We see that $\boldsymbol{x}$ might still be inadmissible for $n \in \{4, 5, 6, 7, 8\}$.

For $k = 2$, the question is whether there exists $n_1 \geq 1$ such that $\Delta_{n_1,2,N} = x_2$. The only candidates for inadmissibility are $x_2 = 4N - D_2$, $D_2 \in \{\lceil N/2^{m-2} \rceil, \lceil N/2^{m-2} \rceil - 1\}$. For the example, $D_2 = \lceil 5/4 \rceil = 2$, or 1, so the candidates are $\Delta_{n_1,2,5} = 20 - D_2 \in \{18, 19\}$. Since $x_2 = 19$, $D_2 = 1$. Thus $I_2 = [1, 3] + 4r_2$, $r_2 = 0, 1, 2, \ldots$ for the range of n accommodating $x_2 = 19$. The values for n tha intersect with those for $k = 1$ are $n \in \{5, 6, 7\}$. So inadmissibility has not been ruled out.

We continue this way examining the x_i, until we finally hit x_{m-1}.

For $k = m - 1$, is there $n_1 \geq 1$ such that $\Delta_{n_1,m-1,N} = x_{m-1}$? The only candidates for inadmissibility are $x_3 = 2^{m-1}N - D_3$, $D_3 \in \{\lceil N/2 \rceil, \lceil N/2 \rceil - 1\}$. For the example, $D_3 = \lceil 5/2 \rceil = 3$, or 2, so the candidates are $\Delta_{n_1,2,5} = 40 - D_3 \in \{37, 38\}$. Since $x_3 = 37$, $D_3 = 3$. Thus $I_3 = [2, 2] + 2r_3$, $r_3 = 0, 1, 2, \ldots$ for the range of n accommodating $x_3 = 37$. We see that the only value intersecting with the above inadmissibility values is $n = 6$. Thus $\boldsymbol{x} = (9, 19, 37, 75) = \boldsymbol{r}_{11} - \boldsymbol{r}_6$ is inadmissible.

Let us now modify the example a bit: $\boldsymbol{x} = (9, 18, 37, 75)$. For $x_2 = 18$ we have $D_2 = 2$, $I_2 = [4, 4] + 4r_2$, so $n \in \{4, 8\}$. These same values intersect with I_3, so the smallest value in the intersection is $n = 4$. Thus, $(9, 18, 37, 75) = \boldsymbol{r}_9 - \boldsymbol{r}_4$ is also inadmissible. Now consider $\boldsymbol{x} = (9, 18, 38, 75)$. For $x_3 = 38$, $D_3 = 2$, $I_3 = [1, 1] + 2r_3$, so $n \in \{1, 3, 5, 7\}$, whose intersection with I_2 for $x_2 = 18$ is empty. Therefore $(9, 18, 38, 75)$ is admissible. However, the intersection of $\{1,3,5,7\}$ with I_2 for $x_2 = 19$ is $\{1,3,5,7\}$, so $n = 5$, and $(9,19,38,75) = \boldsymbol{r}_{10} - \boldsymbol{r}_5$ is inadmissible.

I have not edited beyond this point

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6 A constructive interlude

6.1 The meager rat

We study subtraction games, where both positions and move options are vectors (m-tuples) of non-negative integers, $m \in \mathbb{N}$ a positive integer (here ≥ 2). Let $n \in \mathbb{N}$. The *standard form* for the meager rat is

$$\boldsymbol{r}_2 = \left(\left\lfloor \frac{3}{2} n \right\rfloor, 3n - 1 \right).$$

For $n \in \mathbb{N}$, the meager rat's $\mathcal{P}$*-matrix* give all non-zero P-positions:

$$\mathcal{P}_2 = \begin{pmatrix} 3n-2 & 6n-4 \\ 3n & 6n-1 \end{pmatrix}$$

The meager rat's *forbidden-subtraction matrix* consists of all vector differences of the meager rat's $\mathcal{P}$-matrix, together with the rows in the $\mathcal{P}$ matrix:

$$\overline{\mathcal{S}} = \begin{pmatrix} 3n-2 & 6n-4 \\ 3n & 6n-1 \\ 3n-1 & 6n-3 \\ 3n-2 & 6n-3 \\ 3n & 6n \end{pmatrix}$$

The easiest case in our main result, will be the vector subtracting game, where a player can subtract (s_1, s_2) from a given position (x_1, x_2), unless (s_1, s_2) equals a row in the forbidden-subtraction matrix, for some n. That is, any vector (except $\mathbf{0}$) which is not a row-vector, is a valid subtraction in the game, and provided the resulting position has non-negative coordinates.

Let us say that Lisa plays from the position $(3,7)$. Suppose that she suggests the move $(3,7) - (3,2)$. Before carrying out the move she should answer the following questions (in the mentioned order):

- is the move possible?
- is the move good?

In this case it is easy to check that the move is allowed, because each row in the forbidden-subtraction matrix is increasing. so, indeed, the move to position $(3,7)-(3,2)=(0,5)$ will be carried out. But was this a good move? The answer is in the $\mathcal{P}$-matrix. Since $(0,4)$ is not in this matrix the move was losing. The 'best' move from $(3,7)$ is to subtract $(3,7)$. In fact, the first priority for the current player is to check whether the position belongs to $\overline{\mathcal{S}}$ or not, and it is easy to check that $(3,7)$ does not belong to it; so $(3,7)\to\mathbf{0}$ is a winning move.

Suppose now that the given position is $(5,9)$. It is easy to see that $(5,9)$ is not an allowed subtraction. But neither is $(5,9)$ a P-position. Hence, there is a winning move. In fact, if we look at the nearest P-position, which is $(4,8)$, Lisa can subtract $(1,1)$ and win (albeit not immediately). It is easy to find a winning move for any row-position in this matrix, namely $(1,1)$ and $(0,1)$ respectively. It is also easy to check that, from each position in $\mathcal{P}$, there is no move to $\mathcal{P}$: for example $(3n,6n-1)-(3n-2,6n-4)=(2,3)=(3n-1,6n-3)$, for $n=1$, which is the second row in $\neg\mathcal{S}$. These are the ideas that we used in the proof of the main result, which generalizes the meager rat game to the not so meager rat and the arbitrarily fat rats.

For $n\in\mathbb{Z}_{>0}$, the meager rat's $\mathcal{N}'$*-matrix* (equivalently $\overline{\mathcal{S}}_2\setminus\mathcal{P}_2$ matrix) is:

$$\begin{pmatrix} 3n-1 & 6n-3 \\ 3n & 6n \end{pmatrix}$$

The rows correspond exactly to the N-positions in the game, for which there is no move to $\mathbf{0}$.

6.2 The not so meager rat

Let n run over the positive integers. The standard form for the not so meager rat is

$$\boldsymbol{r}_3=\left(\left\lfloor\frac{7}{4}n\right\rfloor,\left\lfloor\frac{7}{2}n\right\rfloor-1,7n-3\right)$$

Let n run over the nonnegative integers. The not so meager rat matrix is

$$\mathcal{P}_3=\begin{pmatrix} 7n+1 & 14n+2 & 28n+4 \\ 7n+3 & 14n+6 & 28n+11 \\ 7n+5 & 14n+9 & 28n+18 \\ 7n+7 & 14n+13 & 28n+25 \end{pmatrix}$$

$$\overline{\mathcal{S}}_3 \setminus \mathcal{P}_3 = \begin{pmatrix} 7n+1 & 14n+3 & 28n+7 \\ 7n+2 & 14n+3 & 28n+7 \\ 7n+2 & 14n+4 & 28n+7 \\ 7n+3 & 14n+7 & 28n+14 \\ 7n+4 & 14n+7 & 28n+14 \\ 7n+5 & 14n+10 & 28n+21 \\ 7n+5 & 14n+11 & 28n+21 \\ 7n+6 & 14n+11 & 28n+21 \\ 7n+7 & 14n+14 & 28n+28 \end{pmatrix}$$

We code the differences of the entries in this matrix as follows (For example in the first row $1 \times 2 + 1 = 3$ and $3 \times 2 + 1 = 7$.) Thus, in general, it will suffice to understand the first row of $\overline{\mathcal{S}}_m \setminus \mathcal{P}_m$ together with the $\pm$ matrix. But this problem seems hard to resolve.

$$\pm = \begin{pmatrix} + & + \\ - & + \\ 0 & - \\ + & 0 \\ - & 0 \\ 0 & + \\ + & - \\ - & - \end{pmatrix}$$

6.3 The fat rat

Standard form, fat rat

$$\boldsymbol{r}_4 = \left(\left\lfloor \frac{15}{8} n \right\rfloor, \left\lfloor \frac{15}{4} n \right\rfloor - 1, \left\lfloor \frac{15}{2} n \right\rfloor - 3, 15n - 5 \right)$$

The fat rat matrix:

$$\mathcal{P}_4 = \begin{pmatrix} 15n+1 & 30n+2 & 60n+4 & 120n+8 \\ 15n+3 & 30n+6 & 60n+12 & 120n+23 \\ 15n+5 & 30n+10 & 60n+19 & 120n+38 \\ 15n+7 & 30n+14 & 60n+27 & 120n+53 \\ 15n+9 & 30n+17 & 60n+34 & 120n+68 \\ 15n+11 & 30n+21 & 60n+42 & 120n+83 \\ 15n+13 & 30n+25 & 60n+49 & 120n+98 \\ 15n+15 & 30n+29 & 60n+57 & 120n+113 \end{pmatrix}$$

$$\overline{\mathcal{S}}_4 \setminus \mathcal{P}_4 = \begin{pmatrix} 15n+1 & 30n+3 & 60n+7 & 120n+15 \\ 15n+2 & 30n+3 & 60n+7 & 120n+15 \\ 15n+2 & 30n+4 & 60n+7 & 120n+15 \\ 15n+2 & 30n+4 & 60n+8 & 120n+15 \\ 15n+3 & 30n+7 & 60n+15 & 120n+30 \\ 15n+4 & 30n+7 & 60n+15 & 120n+30 \\ 15n+4 & 30n+8 & 60n+15 & 120n+30 \\ 15n+5 & 30n+11 & 60n+22 & 120n+45 \\ 15n+5 & 30n+11 & 60n+23 & 120n+45 \\ 15n+6 & 30n+11 & 60n+22 & 120n+45 \\ 15n+6 & 30n+11 & 60n+23 & 120n+45 \\ 15n+6 & 30n+12 & 60n+23 & 120n+45 \\ 15n+7 & 30n+15 & 60n+30 & 120n+60 \\ 15n+8 & 30n+15 & 60n+30 & 120n+60 \\ 15n+9 & 30n+18 & 60n+37 & 120n+75 \\ 15n+9 & 30n+19 & 60n+37 & 120n+75 \\ 15n+9 & 30n+19 & 60n+38 & 120n+75 \\ 15n+10 & 30n+19 & 60n+37 & 120n+75 \\ 15n+10 & 30n+19 & 60n+38 & 120n+75 \\ 15n+11 & 30n+22 & 60n+45 & 120n+90 \\ 15n+11 & 30n+23 & 60n+45 & 120n+90 \\ 15n+12 & 30n+23 & 60n+45 & 120n+90 \\ 15n+13 & 30n+26 & 60n+52 & 120n+105 \\ 15n+13 & 30n+26 & 60n+53 & 120n+105 \\ 15n+13 & 30n+27 & 60n+53 & 120n+105 \\ 15n+14 & 30n+27 & 60n+53 & 120n+105 \\ 15n+15 & 30n+30 & 60n+60 & 120n+120 \end{pmatrix}$$

$$\pm = \begin{pmatrix} + & + & + \\ - & + & + \\ 0 & - & + \\ 0 & 0 & - \\ + & + & 0 \\ - & + & 0 \\ 0 & - & 0 \\ + & 0 & + \\ + & + & - \\ - & 0 & + \\ - & + & - \\ 0 & - & - \\ + & 0 & 0 \\ - & 0 & 0 \\ 0 & + & + \\ + & - & + \\ + & 0 & - \\ - & - & + \\ - & 0 & - \\ 0 & + & 0 \\ + & - & 0 \\ - & - & 0 \\ 0 & 0 & + \\ 0 & + & - \\ + & - & - \\ - & - & - \end{pmatrix}$$

6.4 Anatomy of the rats

1. Periodicity property. For every $1 \le k \le m$,

$$\left\lfloor \frac{2^m - 1}{2^{m-k}}(n + 2^{m-k}) \right\rfloor = \left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor + 2^m - 1,$$

so

$$\left\lfloor \frac{2^m - 1}{2^{m-k}}(n + 2^{m-k}) \right\rfloor \equiv \left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor \pmod{2^m - 1}.$$

Thus, $\lfloor (2^m - 1)n/2^{m-k} \rfloor$ is periodic mod $2^m - 1$ after 2^{m-k} consecutive values of n.

2. The structure of the $2^{m-1}-1$ row gaps. For $1 \le n < 2^{m-1}$,

$$\Delta_{n,k} := \left\lfloor \frac{2^m-1}{2^{m-k}}(n+1) \right\rfloor - \left\lfloor \frac{2^m-1}{2^{m-k}} n \right\rfloor .$$

For $k = m$, no floors are needed, and $\Delta_{n,m} = 2^m - 1$ for all $n \ge 1$. We may thus assume that $1 \le k < m$. Notice that then $\lfloor (2^m-1)n/2^{m-k} \rfloor = 2^k + \lfloor (-1)/2^{m-k} \rfloor = 2^k - 1$. The floor function basic property implies $2^k - 1 \le \Delta_{n,k} \le 2^k$ for all $n \ge 1$. We next determine for which values of n the gaps assume the value $2^k - 1$, and for which values 2^k is assumed.

The periodicity implies that $\Delta_{n+2^{m-k},k} = \Delta_{n,k}$. Hence it suffices to consider n in the integer interval $I := [1, 2^{m-k}]$. Suppose that for x values n in I the gap 2^k is assumed. Then the gap $2^k - 1$ is assumed for $2^{m-k} - x$ values of n. Thus, $2^k x + (2^k - 1)(2^{m-k} - x) = 2^m - 1$. Solving gives $x = 2^{m-k} - 1$. So only once in I is the gap $2^k - 1$ assumed. Now

$$\Delta_{2^{m-k},k} = \left\lfloor \frac{2^m-1}{2^{m-k}}(2^{m-k}+1) \right\rfloor - \left\lfloor \frac{2^m-1}{2^{m-k}} 2^{m-k} \right\rfloor = 2^k - 1.$$

Thus the extraneous gap is assumed at the end of I. We have proved:

Lemma 6. *$\Delta_{n,k} = 2^k - 1$ for $n \equiv 0 \pmod{2^{m-k}}$; $\Delta_{n,k} = 2^k$ for all other values of $n \ge 1$.*

This lemma gives us a convenient matrix form of the rats' P-positions.

3. The structure of the $m-1$ column gaps.

Lemma 7. *For $1 \le j < m$ and $1 \le i \le 2^{m-1}$, $r_{i,j+1} - r_{i,j} \in \{r_{i,j}, r_{i,j} - 1\}$. Moreover, the binary representation $b(i)$ of i indicates which of the two values is assumed: if $b(i)$ has a 0 in column j, then $r_{i,j+1} - r_{i,j} = r_{i,j}$; if it has a 1 in column j, then $r_{i,j+1} - r_{i,j} = r_{i,j} - 1$.*

Proof. Sketch of proof: From Outline 3, displayed formula for general term of matrix $\mathcal{P}_m$,

$$r_{i,j+1} - r_{i,j} = 2^{j-1} + (i-1)2^j - c_{i,j},$$

where $c_{i,j} = \lceil i/2^{m-j-1} \rceil - \lceil i/2^{m-j} \rceil$, and it remains only to analyze $c_{i,j}$.

6.5 The general matrices

We have two representations for the arbitrarily fat rat. Recall, the m-standard form, using the floor function, for each $m \in \mathbb{Z}_{>1}$:

$$\boldsymbol{r}_n = \left(\left\lfloor \frac{2^m-1}{2^{m-1}} n \right\rfloor, \left\lfloor \frac{2^m-1}{2^{m-2}} n \right\rfloor - 1, \left\lfloor \frac{2^m-3}{2^{m-1}} n \right\rfloor - 3, \ldots, (2^m-1)n - 2^{m-1} + 1 \right),$$

and the matrix form, with $\mu = 2^{m-1}$:

$$\mathcal{P}(m) = \begin{pmatrix} nX_{1,1} + Y_{1,1} & nX_{2,1} + Y_{2,1} & \dots & nX_{\mu,1} + Y_{\mu,1} \\ nX_{1,2} + Y_{1,2} & nX_{2,2} + Y_{2,2} & \dots & nX_{\mu,2} + Y_{\mu,2} \\ \vdots & \vdots & \ddots & \vdots \\ nX_{1,\mu} + Y_{1,\mu} & nX_{2,\mu} + Y_{2,\mu} & \dots & nX_{\mu,\mu} + Y_{\mu,\mu} \end{pmatrix},$$

where

- $X_{i,j}(m) = 2^i(2^m - 1)$,
- $Y_{1,j}(m) = 2j - 1$, and
- $Y_{i,j}(m) = 2Y_{i-1,j}(m) - \zeta_{i,j}$ for $i > 1$,

where $\zeta_{i,j}$ is the jth digit in $(\mu - i)$'s binary representation. The only missing P-position in this matrix representation is $\mathbf{0}$.

6.6 The rat-survivors

The non-reduced $\overline{S}_m$ matrix has $2^{2m-2} + 1$ rows, which may have duplications. Indeed, there are 2^{m-1} P-rows, $2\binom{2^{m-1}}{2}$ subtractions of distinct rows of the P matrix and one subtraction of a P matrix row by itself. (There are 2^{m-1} such self subtractions, but since they are all trivially equal, we take only one.)

Many rows will be identical. We prove that the reduced $\overline{S}_m$ matrix has $3^{m-1} + 2^{m-1}$ rows, as we will now show. For example, the meager rat has 2 P-rows and in total 5 $\overline{S}$-rows, and this number is not possible to reduce. The reduced matrix of the not-so-meager rat has 4 P-rows and 13 altogether (and 17 non-reduced), and so on.

It suffices to show that the reduced $\overline{\mathcal{S}} \setminus \mathcal{P}$ matrix has 3^{m-1} rows, since $\mathcal{P}$ has 2^{m-1} rows. The number of combinations of -,0 and + in the binary representation of the $\overline{\mathcal{S}} \setminus \mathcal{P}$ matrix is 3^{m-1}. Hence it suffices to show that the cancellations of duplicates in all row-differences of the binary matrix corresponds exactly to the cancellations of duplicates in the non-reduced $\overline{\mathcal{S}} \setminus \mathcal{P}$ matrix. This latter matrix is obtained by pairwise subtraction of all rows in the $\mathcal{P}$ matrix. The first column in the $\mathcal{P}$-matrix consists of all odd numbers between 1 and 2^m. Two rows in the ternary matrix are cancelled if and only if each sign is identical. It suffices to prove that standard subtraction of the numbers in binary is identical if and only if the ternary representations are identical. This is obvious.

7 Conclusion

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