When are translations of $\mathcal{P}$-positions of WYTHOFF's game $\mathcal{P}$-positions?

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March 2, 2014

Abstract

We study the problem whether there exist variants of WYTHOFF's game whose $\mathcal{P}$-positions, except for a finite number, are obtained from those of WYTHOFF's game by adding a constant k to each $\mathcal{P}$-position. We solve this question by introducing a class $\{\mathcal{W}_k\}_{k\geq 0}$ of variants of WYTHOFF's game in which, for any fixed $k \geq 0$, the $\mathcal{P}$-positions of $\mathcal{W}_k$ form the set $\{(i,i)|0 \leq i < k\} \cup \{(\lfloor \phi n \rfloor + k, \lfloor \phi^2 n \rfloor + k)|n \geq 0\}$, where ϕ is the golden ratio. We then analyze a class $\{\mathcal{T}_k\}_{k\geq 0}$ of variants of WYTHOFF's game whose members share the same $\mathcal{P}$-positions set $\{(0,0)\} \cup \{(\lfloor \phi n \rfloor + 1, \lfloor \phi^2 n \rfloor + 1)|n \geq 0\}$. We establish several results for the Sprague-Grundy function of these two families. On the way we exhibit a family of games with different rule sets that share the same set of $\mathcal{P}$-positions.

1 Introduction

WYTHOFF's game (WYTHOFF in the sequel), introduced by Willem Abraham Wythoff [19], is a two-pile NIM-like game in which two players move alternately, either removing a number of tokens from one pile, or removing an equal number of tokens from both piles. The player who first cannot move loses, and the opponent wins.

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A position is called an $\mathcal{N}$*-position* (also known as *winning position*) if the $\mathcal{N}$ext player (the player who is about to move from there) can win. Otherwise, the $\mathcal{P}$revious player wins and the position is called a $\mathcal{P}$*-position* (known as *losing position*). Wythoff [19] showed that the $\mathcal{P}$-positions of this game form the set $\{(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) | n \geq 0\}$ in which $\phi = (1 + \sqrt{5})/2$ is the golden ratio and $\lfloor . \rfloor$ denotes the integer part.

Notation 1. For the sake of brevity, we set $A_n = \lfloor \phi n \rfloor$ and $B_n = \lfloor \phi^2 n \rfloor$ for every $n \geq 0$.

Note that $\phi^2 = \phi + 1$, therefore $B_n = A_n + n$.

Several authors have studied variants of WYTHOFF obtained by either adding some extra moves (these new games are known as *extensions*) [5, 8, 7, 9, 11, 14, 15, 16, 17], or eliminating some legal moves (known as *restrictions*) [3, 4, 15]. Variants not of these two typical types have also been widely studied [6, 10, 12, 13]. Not surprisingly, $\mathcal{P}$-positions of these variants are quite diverse.

In this paper we study the question whether there exist variants of WYTHOFF whose $\mathcal{P}$-positions, except possibly some finite number of them, can be obtained by adding a fixed integer $k \geq 1$ to the $\mathcal{P}$-positions of WYTHOFF. More precisely, given an integer $k \geq 1$, we seek non trivial variants of WYTHOFF whose $\mathcal{P}$-positions form a set of the form

$$S \cup \{(A_n + k, B_n + k) | n \geq n_0\}$$

for some $n_0 \geq 0$, where S is a finite set of WYTHOFF's position. Here n_0 can be any nonnegative integer. Below we answer this question for $k \geq 1$.

Recall that the *nim-value* (*Sprague-Grundy value*) of a position is defined inductively as follows: the nim-value of the terminal position (final position) is zero, and the nim-value of a non-terminal position (a, b) is the least integer not in the set of nim-values of the positions directly reachable from (a, b). Note that the set of $\mathcal{P}$-positions of a game is identical to the set of its positions whose nim-values are zero.

In Section 2 we study the family $\{\mathcal{W}_k\}_{k \geq 0}$ of two-pile variants of WYTHOFF in which, for each $\mathcal{W}_k$, each move is one of the following two types:

- (i) removing a number of token from a single pile (*Nim move*), or
- (ii) removing an equal number of tokens from both piles provided that neither of the piles has size less than k after this move (*diagonal move*).

Note that the diagonal move (ii) of $\mathcal{W}_k$ is a constraint on the WYTHOFF move. When $k = 0$, the game $\mathcal{W}_0$ is WYTHOFF. When $k > 0$, one cannot move from (a, b) to $(a - m, b - m)$ if $\min(a - m, b - m) < k$.

We first show that for each $\mathcal{W}_k$, the $\mathcal{P}$-positions form the set

$$\{(i, i)|0 \leq i < k\} \cup \{(A_n + k, B_n + k)|n \geq 0\}.$$

This family of games therefore solves the proposed question. We then explore the sets of those positions whose nim-values are 1 of the family $\{W_k\}_{k\geq 1}$ and, in particular, we prove a recursive relationship between these sets.

Next we examine further modifications of $\{\mathcal{W}_k\}_{k\geq 0}$ in which the diagonal move from each position (a, b), with $a \leq b$, becomes: (ii') removing an equal number i of tokens from both piles such that $a - i \geq k$ and $b - i \geq l$, for some given positive integers k and l with $k \leq l$. We denote this family $\{\mathcal{W}_{k,l}\}$. We prove that the $\mathcal{P}$-positions of the game $\mathcal{W}_{k,l}$, with $k \leq l$, are identical to those of the game $\mathcal{W}_l$. We also formulate a conjecture about an invariance property of the Sprague-Grundy function of members of the family $\{\mathcal{W}_{k,l}\}$.

Section 3 continues the topic of the translation of WYTHOFF's $\mathcal{P}$-positions, studying a variant of WYTHOFF in which the players must consider the integer ratio of the two entries in each diagonal move. Let $k \geq 0$. We analyze a variant of WYTHOFF, called $\mathcal{T}_k$, obtained as follows: from a position (a, b) with $a \leq b$, one can either

- (i) remove a number of tokens from a single pile, or
- (ii) remove an equal number, say s, of tokens from both piles provided that $a - s > 0$ and

$$\left| \left\lfloor \frac{b - s}{a - s} \right\rfloor - \left\lfloor \frac{b}{a} \right\rfloor \right| \leq k.$$

Note that the diagonal move (ii) is a restriction of the diagonal move of WYTHOFF. In this move, the condition $a - s > 0$ guarantees that the ratio $\lfloor (b - s)/(a - s) \rfloor$ is defined. Thus, when making a diagonal move in $\mathcal{T}_k$, one must ensure that the difference between the ratios of the bigger entry over the smaller entry before and after the move must not exceed k.

Consider the special case $k = \infty$. The game $\mathcal{T}_\infty$ is the variant of WYTHOFF in which the only restriction is that the diagonal move cannot make any pile empty.

We now give some examples to illustrate the rule of the game $\mathcal{T}_k$ with some values of k. From the position $(5, 10)$, one can either reduce any single entry, or reduce the same s from both entries provided that $|\lfloor(10-s)/(5-s)\rfloor - \lfloor 10/5\rfloor| \leq k$. Table 1 displays the differences on diagonal moves between the games corresponding to $k = 0, 1, 2, 3, 4$.

k	Original position	s: number of tokens that can be removed in the diagonal move	The options enabled by the diagonal move
0	(5,10)	1, 2	(4,9), (3,8)
1	(5,10)	1, 2, 3	(4,9), (3,8), (2,7)
2	(5,10)	1, 2, 3	(4,9), (3,8), (2,7)
3	(5,10)	1, 2, 3	(4,9), (3,8), (2,7)
4	(5,10)	1, 2, 3, 4	(4,9), (3,8), (2,7), (1,6)

Table 1: Possible diagonal moves from (5,10) for the game with $k = 0, 1, 2, 3$.

We analyze the winning strategy of the game $\mathcal{T}_k$, for given k. We show that the $\mathcal{P}$-positions of game $\mathcal{T}_k$ form the set

$$\{0,0)\} \cup \{(A_n + 1, B_n + 1) | n \geq 0\},$$

which is independent of k. We then study the Sprague-Grundy function of the family $\{\mathcal{T}_k\}_{k\geq 0}$. We prove that all games $\mathcal{T}_k$ share the same positions whose nim-values are 1, forming the set

$$\{(0,1)\} \cup \{(A_n + 2, B_n + 2) | n \geq 0\}.$$

We state a conjecture regarding an invariance property of the nim-value g between two games $\mathcal{T}_k$ and $\mathcal{T}_l$ with $k \leq l$, provided $g \leq k$.

The paper ends with two further questions on the translation of WYTHOFF's $\mathcal{P}$-positions.

2 The class $\mathcal{W}_k$

2.1 The winning strategy

We prove the formula for the $\mathcal{P}$-positions of $\mathcal{W}_k$ in this section. Before doing this, let us recall some background on WYTHOFF. The set of positive integers is denoted by $\mathbb{N}$.

Lemma 1. *[1] The sets* $\{A_n\}_{n\geq 1}$ *and* $\{B_n\}_{n\geq 1}$ *are complementary, namely,*

$$(\cup_{n\geq 1} A_n) \cup (\cup_{n\geq 1} B_n) = \mathbb{N},$$
$$(\cup_{n\geq 1} A_n) \cap (\cup_{n\geq 1} B_n) = \emptyset.$$

Recall that in Wythoff, the following are the $\mathcal{P}$-positions.

Theorem 2. *[19] The* $\mathcal{P}$*-positions of* Wythoff *form the set*

$$\{(A_n, B_n)|n \geq 0\}.$$

We are now able to describe the $\mathcal{P}$-positions for $\mathcal{W}_k$ where $k \geq 1$.

Notation 2. We denote by $\mathcal{S}_k^g$ the set of positions whose nim-values are g in any given game.

Theorem 3. *For each* $k \geq 0$*, the* $\mathcal{P}$*-positions of* $\mathcal{W}_k$ *form the set*

$$\mathcal{S}_k^0 = \{(i,i)|0 \leq i < k\} \cup \{(A_n + k, B_n + k)|n \geq 0\}.$$

Proof. Let

$$\mathcal{A} = \{(i,i)|0 \leq i < k\} \cup \{(A_n + k, B_n + k)|n \geq 0\}.$$

It suffices to verify that the following two properties hold for $\mathcal{W}_k$:

(i) every move from a position in $\mathcal{A}$ cannot terminate in $\mathcal{A}$,

(ii) from every position not in $\mathcal{A}$, there is a move terminating in $\mathcal{A}$.

For (i), note that there is no diagonal move between positions of the form (i,i) where $i \leq k$, by the definition of the game $\mathcal{W}_k$. For $n > 0$ we have $B_n > A_n$, so there is no move from (A_n+k, B_n+k) to (i,i) with $i \leq k, n > 0$. It remains to show that there is no move between two positions of the form (A_n+k, B_n+k). Suppose there is a move $(A_n+k, B_n+k) \to (A_m+k, B_m+k)$ (not necessarily ordered pairs). Then $m < n$. The difference between the amounts taken from the two piles is $n - m > 0$, so this is not a legal move in Wythoff, a fortiori not in $\mathcal{W}_k$.

For (ii), let $(a,b) \notin \mathcal{A}$ with $a \leq b$. If $a \leq k$ then necessarily $b > a$, so reducing b to a leads to a position in $\mathcal{A}$. We now consider the case $k < a \leq b$. If $a = b$ then one can move from (a,b) to $(k,k) \in \mathcal{A}$. It remains to consider the case $k < a < b$. Note that $(a-k, b-k) \notin \{(A_n, B_n)|n \geq 0\}$. By Theorem 2, there exists a move from $(a-k, b-k)$ to some (A_n, B_n) in Wythoff. This implies that there exists a move in $\mathcal{W}_k$ from (a,b) to some $(A_n+k, B_n+k) \in \mathcal{A}$. □

Note that for $k = 0$ the displayed formula for $\mathcal{S}_k^0$ in Theorem 3 gives the $\mathcal{P}$-positions of WYTHOFF, but the proof (at its end) used the known facts about WYTHOFF's $\mathcal{P}$-positions, though it would be easy to avoid this use.

For any set S and term l we define $S + l = \{s + l | s \in S\}$.

Corollary 4. *Let $k \geq 0$ and $l > 0$. The set $\mathcal{S}_{k+l}^0$ of $\mathcal{P}$-positions of the game $\mathcal{W}_{k+l}$ can be given recursively in the form*

$$\mathcal{S}_{k+l}^0 = \{(i, i) | 0 \leq i < l\} \cup \{(a + l, b + l) | (a, b) \in \mathcal{S}_k^0\}.$$

Proof. We have

$$\begin{aligned}
&\{(a + l, b + l) | (a, b) \in \mathcal{S}_k^0\} \\
&= \{(i + l, i + l) | 0 \leq i < k\} \cup \{(A_n + k + l, B_n + k + l) | n \geq 0\} \\
&= \{(i, i) | l \leq i < k + l\} \cup \{(A_n + k + l, B_n + k + l) | n \geq 0\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\{(i, i) | 0 \leq i < l\} \cup \{(i, i) | l \leq i < k + l\} \cup \{(A_n + k + l, B_n + k + l) | n \geq 0\} \\
=&\{(i, i) | 0 \leq i < k + l\} \cup \{(A_n + k + l, B_n + k + l) | n \geq 0\} \\
=&\mathcal{S}_{k+l}^0.
\end{aligned}$$

□

Remark 1. For each k, one may be interested in considering the variant $\mathcal{W}_k'$ of WYTHOFF in which each move is one of the following two types:

(i) removing a number of token from a single pile, or

(ii) removing an equal number of tokens from both piles provided that this move does not lead to a position of the form (i, i) where $i < k$.

Note that in $\mathcal{W}_k'$, one can move to a position of the form (i, j) if $i < j$ and $i < k$. This condition distinguishes the two games $\mathcal{W}_k$ and $\mathcal{W}_k'$. Moreover, $\mathcal{W}_k'$ is an extension of $\mathcal{W}_k$. It is not surprising that the winning strategy for $\mathcal{W}_k'$ is exactly the same to that of $\mathcal{W}_k'$. The proof for the following result is exactly the same as that of Theorem 3.

Theorem 5. *For every $k \geq$, $\mathcal{P}$-positions of $\mathcal{W}_k'$ are identical to those of $\mathcal{W}_k$.*

2.2 The positions with nim-values 1 for $\mathcal{W}_k$

Recall that, for each $k \geq 0$, the set of positions whose nim-values are 1 in the game $\mathcal{W}_k$ is denoted by $\mathcal{S}_k^1$. In this part, we establish $\mathcal{S}_k^1$, for each $k \geq 0$. We first introduce the result for $k = 0$ and $k = 1$. We then show that for $k > 1$, $\mathcal{S}_k^1$ can be derived directly from either $\mathcal{S}_0^1$ (if k is even) or $\mathcal{S}_1^1$ (if k is odd).

For $\mathcal{W}_0$ (WYTHOFF) a recursive algorithm for computing its 1-values was given in [2]. It was conjectured there that the algorithm for computing the n-th 1-value (a_n, b_n) is polynomial in $\Omega(\log n)$. See also [18].

Theorem 6. *The set of positions with nim-value 1 in $\mathcal{W}_1$ is*

$$\mathcal{S}_1^1 = \{(0,1)\} \cup \{(A_n + 2, B_n + 2) | n \geq 0\}.$$

Proof. Set

$$\mathcal{B} = \{(0,1)\} \cup \{(A_n + 2, B_n + 2) | n \geq 0\}.$$

Recall that the set of $\mathcal{P}$-positions of $\mathcal{W}_1$ is

$$\mathcal{S}_1^0 = \{(0,0)\} \cup \{(A_n + 1, B_n + 1) | n \geq 0\}.$$

It suffices to prove the following four facts:

(*i*) $\mathcal{B} \cap \mathcal{S}_1^0 = \emptyset$,

(*ii*) There is no move from a position in $\mathcal{B}$ to a position in $\mathcal{B}$,

(*iii*) From every position in $\mathcal{B}$ there is a move to a position in $\mathcal{S}_1^0$,

(*iv*) From every position not in $\mathcal{B} \cup \mathcal{S}_1^0$, there exists a move to some position in $\mathcal{B}$ (to ensure that $\mathcal{B}$ contains *all* the 1-values).

For (*i*), assume that $\mathcal{B} \cap \mathcal{S}_1^0 \neq \emptyset$. Then there exist $n \neq m$ such that

$$(A_n + 1, B_n + 1) = (A_m + 2, B_m + 2).$$

Then either

$$\begin{cases} A_n = A_m + 1, \\ B_n = B_m + 1, \end{cases}$$

and subtracting gives $n = m$, a contradiction; or else

$$\begin{cases} A_n = B_m + 1, \\ B_n = A_m + 1, \end{cases}$$

and subtracting, gives $n+m=0$ which leads to the contradiction $n=m=0$.

For (ii), it is easy to see that there is no move $(A_n+2, B_n+2) \to (0,1)$. The special case $k=2$ in the proof of (i) in Theorem 3 shows that there is no move between positions of the form (A_n+2, B_n+2).

For (iii), note that in the proof of Theorem 3 we already showed that from every position not in $\mathcal{S}_1^0$ there is a move to a position in $\mathcal{S}_1^0$, so this holds a fortiori for all positions in $\mathcal{B}$ by (i).

For (iv), let $(a,b) \notin \mathcal{B} \cup \mathcal{S}_1^0$ with $a \leq b$. One can move from (a,b) to (0,1) if either $a=0$ or $a=1$ by taking a number of tokens from the pile of size b. If $a \geq 2$, consider the position $p=(a-2, b-2)$. Note that p is not of the form (A_n, B_n). In WYTHOFF, there is a move from p to some position (A_m, B_m). This move results in $(a,b) \to (A_m+2, B_m+2) \in \mathcal{S}_1^1$. □

We now show how $\mathcal{S}_{k+2}^1$ can be obtained from $\mathcal{S}_k^1$.

Theorem 7. *Let $k \geq 0$ be an integer. We have*

$$\mathcal{S}_{k+2}^1 = \{(0,1)\} \cup \{(a+2, b+2) | (a,b) \in \mathcal{S}_k^1\}.$$

Proof. Set

$$\mathcal{C} = \{(0,1)\} \cup \{(a+2, b+2) | (a,b) \in \mathcal{S}_k^1\}.$$

Recall (Theorem 3) that the set of $\mathcal{P}$-positions of $\mathcal{W}_{k+2}$ is

$$\mathcal{S}_{k+2}^0 = \{(i,i) | 0 \leq i < k+2\} \cup \{(A_n+k+2, B_n+k+2) | n \geq 0\}.$$

It suffices to prove the following facts.

(i) $\mathcal{C} \cap \mathcal{S}_{k+2}^0 = \emptyset$.

(ii) There is no move from a position in $\mathcal{C}$ to a position in $\mathcal{C}$.

(iii) From every position not in $\mathcal{C} \cup \mathcal{S}_{k+2}^0$, there exists a move to some position in $\mathcal{C}$.

For (i), note that $(0,1) \notin \mathcal{S}_{k+2}^0$ and so we only need to show that $(A_n + k+2, B_n+k+2) \notin \mathcal{S}_{k+2}^0$. Assume that this is not the case. Then there exists $(a,b) \in \mathcal{S}_k^1$ such that either $(a+2, b+2) = (i,i)$ for some $i < k+2$ or $(a+2, b+2) = (A_n+k+2, B_n+k+2)$ for some $n \geq 0$. It follows from either of these two cases that $(a,b) \in \mathcal{S}_k^0$, a contradiction.

For (ii), we first claim that there is no move from $(a+2, b+2)$ to $(0,1)$ in $\mathcal{W}_{k+2}$. In fact, this move must be diagonal, but since $k+2 \geq 3$, we cannot reach $(0,1)$. We now show that, in $\mathcal{W}_{k+2}$, there is no move between $(a+2,b+2)$ and $(a'+2,b'+2)$ for some (a,b) and (a',b') in $\mathcal{S}_k^1$. In fact, the existence of such a move in $\mathcal{W}_{k+2}$ implies that there exists a move between (a,b) and (a',b') in $\mathcal{S}_k^1$, a contradiction.

For (iii), let $(c,d) \notin \mathcal{C} \cup \mathcal{S}_{k+2}^0$ with $c \leq d$. One can move from (c,d) to (0,1) if either $c=0$ or $c=1$ by taking either $d-1$ or d tokens respectively from the pile of size d. Note that $(0,1) \in \mathcal{S}_k^1$ and so $(2,3) \in \mathcal{C}$. Also note that $(2,2) \in \mathcal{S}_{k+2}^0$. Therefore, if $c=2$, then $d>3$. It follows that one can move from (c,d) to $(2,3) \in \mathcal{S}_{k+2}^1$. We now assume that $c \geq 3$. The position $p=(c-2,d-2) \notin \mathcal{S}_k^1 \cup \mathcal{S}_k^0$: if $p \in \mathcal{S}_k^0$ then $(c,d) \in \mathcal{S}_{k+2}^0$; if $p \in \mathcal{S}_k^1$, then $(c,d) \in \mathcal{C}$. Consequently there exists a move from p to some position $(c',d') \in \mathcal{S}_k^1$. This is equivalent to the fact that there exists a move from (c,d) to the position $(c'+2,d'+2) \in \mathcal{S}_k^1$. This completes the proof. □

Theorems 6 and 7 provide full information on the positions whose nim-values are 1 of the game $\mathcal{W}_k$ when k is odd. Iterating Theorem 7 we get

Corollary 8. *For $k=2l+1$, the positions of the game $\mathcal{W}_k$ whose nim-values are 1 form the set*

$$\{(2i,2i+1)|0 \leq i \leq l\} \cup \{(A_n+k+1, B_n+k+1)|n \geq 0\}.$$

Our computer exploration shows that translation phenomena such as in Theorems 3 and 7 no longer hold for $g \geq 2$. It seems to be hard to get a general formula encompassing all $\mathcal{W}_k$ for the positions whose nim-values are g for some $g \geq 2$.

2.3 An additional generalization

We now investigate a further variant of the game $\mathcal{W}_k$. Let k and l be non-negative integers such that $k \leq l$. We present a variant $\mathcal{W}_{k,l}$ of Wythoff in which each move is one of the following two types:

- (i) removing a number of token from a single pile, or
- (ii) removing an equal number of tokens from both piles provided that the position (i,j) moved to satisfies $\min(i,j) \geq k$ and $\max(i,j) \geq l$.

For example, let $k=3, l=5$. The diagonal move $(6,9) \to (3,6)$ is legal while the move $(6,9) \to (2,5)$ is illegal since $2=\min(2,5)<3$.

Notice that for $k = l$, the rule sets of the games $\mathcal{W}_k$ and $\mathcal{W}_{k,k}$ are identical. The following theorem shows that the $\mathcal{P}$-positions of the game $\mathcal{W}_{k,l}$ depend only on l and, moreover, are identical to those of the game $\mathcal{W}_l$.

Theorem 9. *Let k and l be nonnegative integers with $k \leq l$. The $\mathcal{P}$-positions of $\mathcal{W}_{k,l}$ are identical to those of $\mathcal{W}_l$.*

The proof of Theorem 9 is essentially the same as that of Theorem 3, with l replacing k in Theorem 3. We leave the details to the reader.

We next present a conjecture on the invariance property of the Sprague-Grundy function of $\{\mathcal{W}_{k,l}\}$ implied by our investigations.

Conjecture 10. *Let $k < k' \leq l$. For every integer g in the range $0 \leq g \leq l - k'$, the two games $\mathcal{W}_{k,l}$ and $\mathcal{W}_{k',l}$ have the same sets of positions with nim-value g.*

One may be interested in an investigation on the set of positions whose nim-values are 1 in each game $\mathcal{W}_{k,l}$. By Conjecture 10, we have $\mathcal{S}^v_{k',l} = \mathcal{S}^v_{k,l}$ for $k, k' < l$. Our computer exploration shows that if l is even, the set $\mathcal{S}^v_{k,l}$ seems to be coincident with the set $\mathcal{S}^v_k$. When l is odd, as far as our calculation, the set $\mathcal{S}^v_{k,l}$ is very close to $\mathcal{S}^v_k$, illustrated as follows. Let l is odd and let $\{(a_n, b_n)\}_{n\geq 0}$ (reps. $\{(a'_n, b'_n)\}_{n\geq 0}$) be the sequence of positions of $\mathcal{W}_{k,l}$ (reps. $\mathcal{W}_l$) whose nim-values are 1 such that $a_n \leq b_n$ (reps. $a'_n \leq b'_n$) and $a_i < a_j$ (reps. $a'_i < a'_j$) if $i < j$. Then $|a_n - a'_n| + |b_n - b'_n)| \leq 1$.

3 The class $\mathcal{T}_k$

3.1 The winning strategy

We state and prove the formula for the $\mathcal{P}$-positions of the game $\mathcal{T}_k$ for given k.

Theorem 11. *For each $k \geq 0$, the $\mathcal{P}$-positions of $\mathcal{T}_k$ form the set*

$$\mathcal{S}^0_k = \{(0,0)\} \cup \{(A_n + 1, B_n + 1) | n \geq 0\}.$$

Remark 2. The striking feature of this result is that it is independent of k, quite unlike the result of Theorem 3. In the process of the proof below, the reason for this feature will become clear.

Proof. Let $\mathcal{A} = \{(0,0)\} \cup \{(A_n+1, B_n+1) | n \geq 0\}$. It suffices to show that the following two properties hold for $\mathcal{T}_k$:

(i) every move from a position in $\mathcal{A}$ cannot terminate in $\mathcal{A}$,

(ii) from every position not in $\mathcal{A}$, there is a move terminating in $\mathcal{A}$.

For (i), the requirement $a > s$ implies that for all $k \geq 0$, no diagonal move can be made to $(0,0)$. In particular, the diagonal move $(1,1) \to (0,0)$ cannot be made. Since there is no move between positions of the form (A_n, B_n) in Wythoff and the set of moves in $\mathcal{T}_k$ is a subset of that of Wythoff, there is no move in $\mathcal{T}_k$ between positions of the form (A_n+1, B_n+1).

For (ii), let $p = (a,b)$ be a position not in $\mathcal{A}$. Set $q = (a-1, b-1)$. Then q is not of the form (A_n, B_n). Since there exists a legal move from q to some (A_n, B_n) in Wythoff, there exists a move from p to (A_n+1, B_n+1) in $\mathcal{T}_k$, provided that if a diagonal move is taken then $|\lfloor (B_n+1)/(A_n+1) \rfloor - \lfloor b/a \rfloor| \leq k$. In fact, we now show that $\lfloor (B_n+1)/(A_n+1) \rfloor = \lfloor b/a \rfloor$, so the inequality holds for all k. This explains why the expression for the $\mathcal{P}$-positions is independent of k: Since $(a-1, b-1) \to (A_n, B_n)$ is also a diagonal move in Wythoff, the move must satisfy $(b-1)-(a-1) = b-a = B_n - A_n = n$. Now $(B_n+1)/(A_n+1) = (A_n+n+1)/(A_n+1) = 1+n/(A_n+1)$. Since $\phi > 1$, $n < A_n+1$, so $\lfloor (B_n+1)/(A_n+1) \rfloor = 1$. Also $b/a = (a+n)/a = 1+n/a$. If $a \to A_n$, then $n \leq A_n < a$, and if $a \to B_n$, then $n \leq A_n \leq B_n < a$, so in either case $n < a$. Hence $\lfloor b/a \rfloor = \lfloor (B_n+1)/(A_n+1) \rfloor = 1$. □

A comparison between Theorems 3 and 11 immediately implies:

Corollary 12. *The set of $\mathcal{P}$-positions of $\mathcal{W}_1$ is identical to the set of $\mathcal{P}$-positions of $\mathcal{T}_k$ for every $k \geq 0$.*

It is rather rare that two games with different rule-sets have the same set of $\mathcal{P}$-positions. In fact, we have here a family of games that share the same $\mathcal{P}$-positions, since the rule sets of $\mathcal{T}_k$ are different for each k. Why does it happen here? Why is each $\mathcal{P}$-position of $\mathcal{T}_k$ but a translation by 1 of a $\mathcal{P}$-position of Wythoff (except for $(0,0)$)? We end this section with some intuition about these questions.

Consider the game $\mathcal{T}_\infty$. This is the same as Wythoff, except that the terminal position $(0,0)$ of Wythoff cannot be reached with a diagonal move. The position $(1,1)$ is terminal in $\mathcal{T}_\infty$ – for diagonal moves – and so replaces the terminal position $(0,0)$ of Wythoff. In the proof of (ii)

of Theorem 11, the reason of the independence of the $\mathcal{P}$-positions of k was explained. This independence includes the case $k = \infty$. Thus the $\mathcal{P}$-positions (A_n, B_n) of WYTHOFF are translated into the $\mathcal{P}$-positions $(A_n + 1, B_n + 1)$ in $\mathcal{T}_k$. Note further that the rule-sets of $\mathcal{W}_1$ and $\mathcal{T}_\infty$ are identical. Therefore the two games have identical sets of $\mathcal{P}$-positions. But as pointed out in the previous paragraph, we indeed have an entire family of different rule-sets sharing the same set of $\mathcal{P}$-positions.

3.2 On an invariance property of nim-values

We state two properties of the Sprague-Grundy function of the class of games $\mathcal{T}_k$. First, the games $\mathcal{T}_k$, for different values k, share the same set of positions with nim-values 1. It is then conjectured that, for given k and l, the two games $\mathcal{T}_k$ and $\mathcal{T}_l$ have the same sets of positions of nim-value g, provided that $g \leq \min(k, l)$.

Theorem 13. *For all $k \geq 0$, the set of positions with nim-value 1 in $\mathcal{T}_k$ is*

$$\mathcal{S}_k^1 = \{(0, 1)\} \cup \{(A_n + 2, B_n + 2) | n \geq 0\}.$$

Note 1. This set is identical with $\mathcal{S}_1^1$ for $\mathcal{W}_1$ of Theorem 6.

Proof. As in the proof of Theorem 6, put $\mathcal{B} = \{(0, 1)\} \cup \{(A_n+2, B_n+2) | n \geq 0\}$. Recall that the set of $\mathcal{P}$-positions of $\mathcal{T}_k$ is

$$\mathcal{S}_k^0 = \{(0, 0)\} \cup \{(A_n + 1, B_n + 1) | n \geq 0\}.$$

Analogously to the proof of Theorem 6, it suffices to prove the following properties:

- (i) $\mathcal{B} \cap \mathcal{S}_k^0 = \emptyset$,
- (ii) there is no move from a position in $\mathcal{B}$ to a position in $\mathcal{B}$,
- (iii) from every position in $\mathcal{B}$ there is a move to $\mathcal{S}_k^0$.
- (iv) from every position not in $\mathcal{B} \cup \mathcal{S}_k^0$, there exists a move to some position in $\mathcal{B}$.

As pointed out at the end of section 3.1, the games $\mathcal{T}_\infty$ and W_1 are identical. By Theorem 6, $\mathcal{B} = S_1^1$. Therefore, (i) holds. For (ii), note that the set of moves in $\mathcal{T}_k$ is a subset of that in WYTHOFF. As there is no move between positions of the form (A_n, B_n) in WYTHOFF, there is no move between positions of the form $(A_n + 2, B_n + 2)$ in $\mathcal{B}$. Item (iii) follows a fortiori from (ii) in the proof of Theorem 11. Finally, (iv) holds for all k since each $\mathcal{T}_k$ is an extension of $\mathcal{T}_\infty$. □

Denote by $\mathcal{S}_k^g(\mathcal{T})$ the set of positions with nim-value g in $\mathcal{T}_k$ and by $\mathcal{S}_k^g(\mathcal{W})$ the set of positions with nim-value g in $\mathcal{W}_k$. Corollary 12 states that $\mathcal{S}_1^0(\mathcal{W}) = \mathcal{T}_k^0(\mathcal{T})$ for all $k \geq 0$; Theorems 6 and 13 show that $\mathcal{S}_1^1(\mathcal{W}) = \mathcal{S}_k^1(\mathcal{T})$ for all $k \geq 0$. These results seem to suggest that $S_1^g(\mathcal{W}) = \mathcal{S}_k^g(\mathcal{T})$ for all $g \geq 0$ and all $k \geq 0$. However, there are counterexamples. Thus, for the position $(20, 30)$ we have $g(20, 30) = 38$ in $\mathcal{W}_1$, but $g(20, 30) = 2$ in $\mathcal{T}_1$. Since it seems, however, that $g(20, 30) = 38$ in $\mathcal{T}_k$ for all $k \geq 38$, we are led to the following

Conjecture 14. *Let k be a nonnegative integer. Then*

- $\mathcal{S}_k^g(\mathcal{T}) = \mathcal{S}_\infty^g(T)$ *for all* $0 \leq g \leq k$.
- $\mathcal{S}_1^g(\mathcal{W}) = \mathcal{S}_k^g(\mathcal{T})$ *for all* $0 \leq g \leq k$.

Here is a related

Conjecture 15. *Let k and l be nonnegative integers. For every integer g in the range $0 \leq g \leq \min(k, l)$, we have $\mathcal{S}_k^g = \mathcal{S}_l^g$.*

Note that Conjectures 15, 14 and 10 are related. We believe that a proof method of either of them would lead to the proof of the other two.

4 Conclusion

We found cases when translations of $\mathcal{P}$-positions of WYTHOFF's $\mathcal{P}$-positions are $\mathcal{P}$-positions of games "close" to WYTHOFF. There are some further directions of study on the theme of this translation. We list here two such questions.

Question 16. *Does there exist a variant of* WYTHOFF *whose $\mathcal{P}$-positions, except possibly a finite number, are $(A_n - k, B_n - k)$ for some fixed $k \geq 1$?*

More generally,

Question 17. *Does there exist a variant of* WYTHOFF *whose $\mathcal{P}$-positions, except possibly a finite number, are $(A_n + k, B_n + l)$ for some fixed integers $k \neq l$?*

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