Variants of (s,t)-Wythoff's game [1]

Haiyan Li[2], Sanyang Liu

School of Mathematics and Statistics,
Xidian University, Xi'an, 710126, China

Aviezri S. Fraenkel,

Dept. of Computer Science and Applied Mathematics,
Weizmann Institute of Science, Rehovot 76100, Israel

Wen An Liu

College of Mathematics and Information Science,
Henan Normal University, Xinxiang, 453007, China

Abstract

In this paper, we study four games, they are all restrictions of (s,t)-Wythoff's game which was introduced by A.S. Fraenkel. The first one is a modular type restriction of (s,t)-Wythoff's game, where a player is restricted to remove a multiple of K tokens in each move (K is a fixed positive integer). The others we called rook type restrictions of (s,t)-Wythoff's game, including Odd-Arbitrary-Nim (s,t)-Wythoff's Game, Odd-Odd-Nim (s,t)-Wythoff's Game and Odd-Even-Nim (s,t)-Wythoff's Game. In these three games, the restrictions are only made on horizontal and vertical moves, but not on the extended diagonal moves. For any $K, s, t \geq 1$, the sets of P-positions of our games are given in both normal and misère play.

Keywords P-position; (s,t)-Wythoff's game; restriction; normal play convention

1 Introduction

Introduced by A.S. Fraenkel in [5], (s,t)-Wythoff's game is a well-known 2-player combinatorial game involving two piles of finitely many tokens. Given two integers $s, t \geq 1$, a player may either remove any positive number of tokens from a single pile or remove tokens from both piles, $k > 0$

[1] This work was supported by the National Natural Science Foundation of China (Grants No. 61373174).

[2] Corresponding author. E-mail address: lihaiyan1107@126.com(Haiyan Li).

from one pile and $\ell > 0$ from the other, say $\ell \geq k$, constrained by

$$0 \leq \ell - k < (s-1)k + t. \tag{1}$$

In normal play, the player first unable to move loses; while in misère play that player wins.

The special case $s = t = 1$ is the classical Wythoff's game, while the case $s = 1$, $t \geq 1$ is Generalized Wythoff [4]. More variants of Wythoff's game and (s,t)-Wythoff's game can be found in [2, 3, 11, 12, 14, 15]. For more theory of general combinatorial games, see [1, 6, 7, 10].

By (a,b) we denote a game position with the two piles of sizes a and b. A position is called an N-position (known as winning position) from which the *Next* player can win. Otherwise, it is a P-position (known as losing position) from which the *Previous* player has a winning strategy. We denote by $\mathscr{P}$ and $\mathscr{N}$ the set of all P-positions of a game and the set of all its N-positions respectively. By $\mathbb{Z}^0$ and $\mathbb{Z}^+$ we denote the set of nonnegative integers and positive integers respectively.

Given any game, we notice that the set of all its P-positions constitutes an independent set, and the main goal is to find characterizations of the sequence of P positions. For example, in [5], the author gave all P-positions of (s,t)-Wythoff's game in normal play:

$$\mathscr{P} = \bigcup_{n=0}^{\infty} \{(A'_n, B'_n),\ A'_n = \operatorname{mex}\{A'_i, B'_i \mid 0 \leq i < n\},\ B'_n = sA'_n + tn, \tag{2}$$

where $\operatorname{mex} S = \min(\mathbb{Z}^0 \setminus S)$. In particular, $\operatorname{mex} \emptyset = 0$. In misère play, the set of all P-positions of (s,t)-Wythoff's game was determined in [13].

All four games in this paper are 2-player games played on two piles of finitely many tokens. Let

$$K \in \mathbb{Z}^+, \quad \mathcal{M}_K = \{nK \mid n \in \mathbb{Z}^0\}.$$

Now we define the first game which is a *modular type restriction* of (s,t)-Wythoff's game, denoted by Γ_K: Let $K, s, t \in \mathbb{Z}^+$, a player may either

I. remove $0 < k \in \mathcal{M}_K$ tokens from a single pile, or

II. remove from both piles, $0 < k \in \mathcal{M}_K$ tokens from one pile and $0 < \ell < \mathcal{M}_K$ from the other, subject to the constraint (1).

Notice that the case $K = 1$ is exactly (s,t)-Wythoff's game, while for $K = 2$, it is the "Even Even" case studied in [12].

The remaining three games are called Odd-Arbitrary-Nim (s,t)-Wythoff's Game, Odd-Odd-Nim (s,t)-Wythoff's Game and Odd-Even-Nim (s,t)-Wythoff's Game. These games are *rook type restrictions* of (s,t)-Wythoff's game, which are denoted by Γ_{OA}, Γ_{OO}, Γ_{OE}, respectively. Throughout play of each of these three games, one pile is "first pile" and the other "second pile". In general, we denote by (x,y) a game position where x and y are the numbers of tokens in the first and the second pile, respectively.

(1). In Γ_{OA}, a player may either remove an *odd* number $k > 0$ of tokens from the first pile or an arbitrary number of tokens from the second pile, or move from both piles as in (s,t)-Wythoff's game.

(2). In Γ_{OO}, a player may only remove an *odd* number $k > 0$ of tokens when moving from a single pile (either the first or the second), while the move rule when moving from both piles is the same as that of (s,t)-Wythoff's game.

(3). In Γ_{OE}, a player may either remove an *odd* number $k > 0$ of tokens from the first pile or an *even* number $\ell > 0$ of tokens from the second, or move from both piles as in (s,t)-Wythoff's game.

Notice that in these three games no restriction is imposed on the diagonal move, while for Γ_K and the games defined in [12] also the diagonal move is constrained.

Section 2 provides methods for finding the P-positions of a game and its winning strategy. In Section 3, all P-positions of Γ_K are given recursively in terms of the mex function in both normal and misère play (Theorems 3 and 6). Moreover, a poly-time winning strategy for Γ_K in normal play is provided by exhibiting a relationship between Γ_K and (s,t)-Wythoff's game (Theorem 4 and Corollary 5), together with a special numeration system. While in misère play, a poly-time winning strategy for Γ_K is provided when $s = 1$ (Theorem 7 and Corollary 8). All P-positions of Γ_{OA}, Γ_{OO}, Γ_{OE} in both normal and misère play are given in Section 4 (Theorems 9-16), based on algebraic structures, which provide polynomial time strategies. The final section 5 lists several far-reaching relevant open problems.

2 Preliminaries

It follows from the definition of P- and N-positions that from any N-position there always exists a move to a P-position and from a P-position a player can only move to an N-position (i.e., there can never be a move from a P-position to another P-position). These properties can be used to check whether a given position (a,b) is a P-position or not. By $F(u)$ we denote the *followers* of u, i.e., all positions that can be reached from u in one legal move. Symmetry of the game rules of Γ_K implies that both (a,b) and (b,a) are P-positions (or N-positions). For convenience, however, we agree to write (a,b) with $a \le b$ throughout.

Example 1. For $K = s = 2$ and $t = 1$, consider Γ_K in normal play. We proceed according to the following steps to determine the first few P- and N-positions:

Step 1 P-positions: Clearly, $(0,0),(0,1),(1,1) \in \mathscr{P}$, since the next player has no legal move from them and loses, that is, the previous player wins by default.

Step 2 N-positions: For $(0,m),(1,m),(m,m),(m,m+1),(m,m+2)$ with $m \ge 2$ and $(m,m+3)$ with m positive even, it is easy to check that from each of them a legal move of type I or II can result in a position in $\{(0,0),(0,1),(1,1)\}$, thus they are all N-positions.

Step 3 P-positions: $F(2,6) = \{(0,2),(0,4),(0,6),(2,2),(2,4)\}$. It follows from Step 2 that each position of $F(2,6)$ is an N-position. Thus $(2,6) \in \mathscr{P}$. In the same manner, we can obtain that $(2,7),(3,6),(3,7) \in \mathscr{P}$.

By repeating Steps 2 and 3, we can get more P-positions and N-positions of Γ_K.

3 Modular type restriction of (s,t)-Wythoff's game

We denote by $\lfloor x \rfloor$ the largest integer $\leq x$ and $\lceil x \rceil$ the smallest integer $\geq x$. By $\mathbb{Z}^{\geq m}$ we denote the set of all integers not less than m.

Definition 1. (i) For any set E and any element w, we define $E + w = \{e + w \mid e \in E\}$. In particular, $E = \emptyset \implies E + w = \emptyset$.

(ii) Let $K, s, t \in \mathbb{Z}^+$, and $\Omega_K = \{0, 1, 2, \ldots, K-1\}$. We define two sequences A_n and B_n, for $n \in \mathbb{Z}^0$:

$$\begin{cases} A_n = \operatorname{mex}\{\{A_i \mid 0 \leq i < n\} + \alpha, \{B_i \mid 0 \leq i < n\} + \beta\}, \text{ where } \alpha, \beta \in \Omega_K, \\ B_n = sA_n + \lceil t/K \rceil Kn. \end{cases} \tag{3}$$

Notice that For $K = 1$, $A_n = A'_n$, $B_n = B'_n$, where A'_n, B'_n were defined in Eq. (2).

Lemma 2. *Let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ be defined by Eq. (3). We have the following properties:*

(a) $A_n, B_n \in \mathcal{M}_K$, for $n \in \mathbb{Z}^0$.

(b) For every $n > m \geq 0$, we have $B_n > A_n > A_m$.

(c) Let $A = \bigcup_{n=1}^{\infty}\{A_n\} + \alpha$ and $B = \bigcup_{n=1}^{\infty}\{B_n\} + \beta$, with $\alpha, \beta \in \Omega_K$. Then A and B are complementary with respect to $\mathbb{Z}^{\geq K}$, i.e., $A \cup B = \mathbb{Z}^{\geq K}$ and $A \cap B = \emptyset$.

(d) $A_n - A_{n-1} \in \{K, 2K\}$.

(e) $B_n - B_{n-1} \in \{sK + \lceil t/K \rceil K, 2sK + \lceil t/K \rceil K\}$. Moreover, $B_n - B_{n-1} = sK + \lceil t/K \rceil K$ if and only if $A_n - A_{n-1} = K$; $B_n - B_{n-1} = 2sK + \lceil t/K \rceil K$ if and only if $A_n - A_{n-1} = 2K$.

Proof. (a) Induction on n. Obviously, $A_0 = B_0 = 0$, $A_1 = K$ and $B_1 = sA_1 + \lceil t/K \rceil K \in \mathcal{M}_K$. Suppose $A_j, B_j \in \mathcal{M}_K$ holds for all $j < n$. We now show that $A_n \in \mathcal{M}_K$, and so $B_n = sA_n + \lceil t/K \rceil Kn \in \mathcal{M}_K$.

Indeed, suppose that there exists some $q \in \mathbb{Z}^0$ such that $A_n = qK + \gamma$ with $0 < \gamma \in \Omega_K$. Let $S = \{\{A_i \mid 0 \leq i < n\} + \alpha, \{B_i \mid 0 \leq i < n\} + \beta\}$ with $\alpha, \beta \in \Omega_K$. Then we have $qK + \gamma = \operatorname{mex} S$. This implies that $qK + \gamma \notin S$ and $qK = A_n - \gamma \in S$. If there exist $i_0 < n$ and α, $\beta \in \Omega_K$ such that $qK = A_{i_0} + \alpha$ or $qK = B_{i_0} + \beta$, then by assumption $A_{i_0}, B_{i_0} \in \mathcal{M}_K$ implying that $\alpha = \beta = 0$. Hence $qK + \gamma = A_{i_0} + \gamma \in S$ or $qK + \gamma = B_{i_0} + \gamma \in S$, giving a contradiction.

(b) A_n and B_n are strictly increasing sequences, which is obvious from their definition, and $B_n = sA_n + \lceil t/K \rceil Kn \geq A_n + Kn > A_n > A_m$, for any $n > m \geq 0$.

(c) It is easy to see that $A \cup B = \mathbb{Z}^{\geq K}$. Suppose $A \cap B \neq \emptyset$. It follows from (a) that $A_m + \alpha' \neq B_n$ and $A_m \neq B_n + \beta'$ with $\alpha' > 0$, $\beta' > 0$, thus the only possibility is $A_m = B_n$ for some integers $m, n \in \mathbb{Z}^+$. If $m > n$, then A_m is mex of a set containing $B_n = A_m$, a contradiction. If $m \leq n$, then by (b) we have $B_n = sA_n + \lceil t/K \rceil Kn \geq sA_m + \lceil t/K \rceil Km > A_m$, another contradiction.

(d) By (a) and (b), $0 < A_n - A_{n-1} \in \mathcal{M}_K$. Assume that $A_n - A_{n-1} \geq 3K$, then $A_{n-1} < A_{n-1} + K < A_{n-1} + 2K < A_{n-1} + 3K \leq A_n$. By (c), $A_{n-1} + \omega \in S$ with $1 \leq \omega \leq 3K - 1$. Further, the only possibility is that $A_{n-1} + \omega \in B$. Since $A_n, B_n \in \mathcal{M}_K$, there exists some $j < n$ such that $A_{n-1} + K = B_j$ and $A_{n-1} + 2K = B_{j+1}$. Hence, we get $K = B_{j+1} - B_j = s(A_{j+1} - A_j) + \lceil t/K \rceil K > K$, a contradiction.

(e) Directly from the definition of B_n and (d). ■

Theorem 3. *Let $K, s, t \in \mathbb{Z}^+$. For Γ_K in normal play,*

$$\mathscr{P} = \bigcup_{n=0}^{\infty} \{(A_n + \alpha, B_n + \beta) \mid \alpha, \beta \in \Omega_K\},$$

where A_n and B_n are defined in (3).

Proof. It evidently suffices to show two things:

Fact I. (stability property). No followers of a position in $\mathscr{P}$ can be in $\mathscr{P}$.

Fact II. (absorbing property). Any position not in $\mathscr{P}$ can land in a position in $\mathscr{P}$.

Proof of Fact I. Let (x, y) with $x \leq y$ be a position in $\mathscr{P}$. Clear for $(x, y) \in \Omega_K \times \Omega_K$. For $x, y \geq K$, it follows from Lemma 2(c) that there exist some $n \in \mathbb{Z}^+$ and $\alpha, \beta \in \Omega_K$ such that $(x, y) = (A_n + \alpha, B_n + \beta)$.

It is obvious that a type I move from (x, y) leads to a position not in $\mathscr{P}$. Suppose that $(x, y) \to (x', y') \in \mathscr{P}$ by a type II move. By Lemma 2(a) and (b), there exists m $(< n)$ such that $k = A_n - A_m \in \mathcal{M}_K$ and $\ell = B_n - B_m \in \mathcal{M}_K$. Note that $\lceil t/K \rceil K \geq t$ for any $K, t \in \mathbb{Z}^+$, thus $0 < k \leq \ell = s(A_n - A_m) + \lceil t/K \rceil K(n - m) \geq sk + t$, which contradicts Eq. (1).

Proof of Fact II. Let (x, y) with $x \leq y$ be a position not in $\mathscr{P}$. If $x \in \Omega_K$, let $y = qK + \beta$, $q \in \mathbb{Z}^+$ and $\beta \in \Omega_K$, then move $y \to \beta$. If $x \geq K$, from Lemma 2(c), we have either $x = B_n + \beta$ or $x = A_n + \alpha$ for some $n \in \mathbb{Z}^+$ and $\alpha, \beta \in \Omega_K$.

Case (i) $x = B_n + \beta$. Let $y = qK + \alpha$, $q \in \mathbb{Z}^0$ and $\alpha \in \Omega_K$, we move $y \to A_n + \alpha$, since $y \geq x = B_n + \beta \geq B_n > A_n + \alpha$ and $y - A_n - \alpha \in \mathcal{M}_K$.

Case (ii) $x = A_n + \alpha$. In this case, let $y = qK + \beta$, $q \in \mathbb{Z}^0$, $\beta \in \Omega_K$. We proceed by distinguishing three subcases: (ii.1) $y > B_n + K - 1$, (ii.2) $x \leq y < sA_n + \lceil t/K \rceil K$, (ii.3) $sA_n + \lceil t/K \rceil K \leq y < B_n$.

(ii.1) $y > B_n + K - 1$. Then move $y \to B_n + \beta$.

(ii.2) $x \le y < sA_n + \lceil t/K \rceil K$. We move $(x, y) \to (\alpha, \beta)$. This move is legal: (a) $0 < k = A_n \in \mathcal{M}_K$, (b) $0 < \ell = y - \beta \in \mathcal{M}_K$, (c) $\ell - k = y - \beta - A_n \le (s-1)A_n + \lceil t/K \rceil K - K < (s-1)k + t$.

(ii.3) $sA_n + \lceil t/K \rceil K \le y < B_n$. Put $m = \lfloor (y - sA_n - \beta)/(\lceil t/K \rceil K) \rfloor$. Then move $(x, y) \to (A_m + \alpha, B_m + \beta)$. This move is legal:

(a) $0 < k \in \mathcal{M}_K$. We first prove $0 \le m < n$. Since $y - sA_n \ge \lceil t/K \rceil K \ge K > \beta$, then $(y - sA_n - \beta)/(\lceil t/K \rceil K) > 0$, and so $m = \lfloor (y - sA_n - \beta)/(\lceil t/K \rceil K) \rfloor \ge 0$. On the other hand, $y - sA_n - \beta < B_n - sA_n = \lceil t/K \rceil Kn$, thus $m = \lfloor (y - sA_n - \beta)/(\lceil t/K \rceil K) \rfloor \le (y - sA_n - \beta)/(\lceil t/K \rceil K) < n$. Hence $k = A_n - A_m > 0$.

(b) $0 < \ell \in \mathcal{M}_K$. We know $m \le (y - sA_n - \beta)/(\lceil t/K \rceil K)$, it follows that $y \ge \lceil t/K \rceil Km + sA_n + \beta = B_m + \beta + s(A_n - A_m) > B_m + \beta$. Thus $\ell = y - B_m - \beta > 0$ and clearly $\ell \in \mathcal{M}_K$.

(c) $k \le \ell < sk + t$. By the definition of m, we have $m > (y - sA_n - \beta)/(\lceil t/K \rceil K) - 1$, then $y < \lceil t/K \rceil K(m+1) + sA_n + \beta$. Thus $y - B_m - \beta < s(A_n - A_m) + \lceil t/K \rceil K$. Further, $y - B_m - \beta \le s(A_n - A_m) + \lceil t/K \rceil K - K < s(A_n - A_m) + t$. On the other hand, by (b), $y - B_m - \beta \ge s(A_n - A_m) \ge A_n - A_m$. ■

Theorem 3 provides a recursive winning strategy which is exponential in the input size $\log xy$ of any game position $(x, y) \in \mathbb{Z}^0 \times \mathbb{Z}^0$.

For every $n \in \mathbb{Z}^0$, the pair (A_n, B_n) is called a P-*generator* of P-positions, since the pair generates the set $\{(A_n + \alpha, B_n + \beta) \mid \alpha, \beta \in \Omega_K\}$ of P-positions.

Now the original (s, t)-Wythoff's game with parameters $s, t \in \mathbb{Z}^+$ is the case $K = 1$ of Γ_K. Its P-positions are exactly those in Eq. (2). With Γ_K, $K > 1$, we *associate* an (s, t')-Wythoff's game $\Gamma := \Gamma_1$ with parameters $s(\Gamma) = s(\Gamma_K)$, $t'(\Gamma) = \lceil t/K \rceil$, K as in Γ_K.

In order to provide a poly-time winning strategy for Γ_K, we next exhibit a simple relationship between the P-generators of Γ_K and the P-positions of the associated Γ, which are those of (2), but with t replaced by t':

Theorem 4. *$A'_n = A_n/K$, $B'_n = B_n/K$, where $\{(A_n, B_n)\}_{n \ge 0}$ and $\{(A'_n, B'_n)\}_{n \ge 0}$ are the P-generators of Γ_K and the P-positions of Γ respectively.*

Example 2. For $K = 3$, $s = 2$, $t \in \{4, 5, 6\}$, we display the first few P-generators of Γ_3 and the first few P-positions of the associated Γ in Tables 1 and 2. Notice the divisibility enunciated by Theorem 4.

Table 1: The first few P-generators of Γ_3.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A_n	0	3	6	9	15	18	21	27	30	33	39	42	45	48
B_n	0	12	24	36	54	66	78	96	108	120	138	150	162	174

Table 2: The first few P-positions of the associated Γ.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A'_n	0	1	2	3	5	6	7	9	10	11	13	14	15	16
B'_n	0	4	8	12	18	22	26	32	36	40	46	50	54	58

Proof. From Lemma 2, for all $n \geq 0$: (i) $A_n, B_n \in \mathcal{M}_K$, (ii) $A_{n+1} - A_n \in \{K, 2K\}$, (iii) $B_{n+1} - B_n \in \{sK + \lceil t/K \rceil K, 2sK + \lceil t/K \rceil K\}$.

We see, in particular, that $A_n/K, B_n/K$ are nonnegative integers.

From the proof of Theorem 3.1 of [5] we have: (i)' $A'_{n+1} - A'_n \in \{1, 2\}$, (ii)' $B'_{n+1} - B'_n \in \{s + t', 2s + t'\}$.

(i)', (ii)' follow from (ii), (iii) respectively by dividing by K. But the theorem is not yet proved: it could presumably happen, for example, that for some $n \geq 0$, $A_{n+1} - A_n = 2K$, yet $A'_{n+1} - A'_n = 1$ rather than 2. We now show, however, by induction on n, that

$$(A_{n+1} - A_n)/K = A'_{n+1} - A'_n, \; (B_{n+1} - B_n)/K = B'_{n+1} - B'_n \tag{4}$$

for all $n \geq 0$. The theorem's assertion clearly holds for $n = 0$. Further, from the definition of A_n, B_n we get: $A_1 = K$, $B_1 = sK + \lceil t/K \rceil K$; and from the definition of A'_n, B'_n: $A'_1 = 1$, $B'_1 = s+t'$. Thus Eq. (4) holds for $n = 0$. Suppose $(A_{j+1}-A_j)/K = A'_{j+1}-A'_j$, $(B_{j+1}-B_j)/K = B'_{j+1} - B'_j$ holds for all $j < n$. If $A_{n+1} = A_n + K$, it follows from the mex function and the induction hypothesis that $A'_{n+1} = A'_n + 1$. Similarly, $A_{n+1} = A_n + 2K$ implies $A'_{n+1} = A'_n + 2$. Also B_{n+1}, B'_{n+1} are uniquely determined by A_{n+1}, A'_{n+1} respectively. Thus, again by the induction hypothesis (on A_n, A'_n), Eq. (4) is established, so the theorem's assertion follows. ■

Corollary 5. *In normal play, (x, y) is a P-position of Γ_K if and only if $(\lfloor x/K \rfloor, \lfloor y/K \rfloor)$ is a P-position of Γ.*

Proof. If (x, y) is a P-position of Γ_K with its P-generator being (A_{i_0}, B_{i_0}), $i_0 \in \mathbb{Z}^0$, then by Theorem 4, $(\lfloor x/K \rfloor, \lfloor y/K \rfloor) = (A'_{i_0}, B'_{i_0})$, and vice versa. ■

We now show how Theorem 4 leads to a poly-time winning strategy for Γ_K. Let $u_{-1} = 1/s$, $u_0 = 1$, $u_n = (s + t' - 1)u_{n-1} + su_{n-2}$ $(n \geq 1)$. Denote by $\mathcal{U}$ the numeration system with bases u_0, $u_1, \ldots$ and digits $d_i \in \{0, \ldots, s + t' - 1\}$ such that $d_{i+1} = s + t' - 1 \implies d_i < s$ $(i \geq 0)$. In [5] it was shown (as a special case of a somewhat more general numeration system) that every positive integer N has a unique representation $R(N)$ over $\mathcal{U}$.

The *vile* numbers are those whose representations $R(N)$ end in an even number of 0s,and the *dopey* numbers are those whose representations end in an odd number of 0s. (For an explanation/etymology of the terms vile, dopey, see [8].) Also, y is a *left shift* of x, if $R(y)$

is obtained from $R(x)$ by adjoining 0 to the right end of $R(x)$. Thus, in binary, the decimal number 6 is a left shift of the decimal 3, since $R(6) = 110$, $R(3) = 11$; 3 is vile since $R(3)$ ends in an even number (zero) of 0s and 6 is dopey.

In [5] it was proved that $(x, y) \in \Gamma$ with $x \le y$ is a P-position of Γ if and only if x is vile and y is a left shift of x (so it is dopey). The fact that the u_i grow exponentially, together with Theorem 4 clearly provide a poly-time winning strategy for Γ_K. For $K = 2$ this provides a poly-time winning strategy for the "Even Even" case, which remained elusive in [12].

Notice that if s, t are the parameters of Γ_K, then s, t' are the parameters of Γ, where $t' = \lceil t/K \rceil$.

Example 3. Consider Γ_3 of Example 2, where $K = 3$, $s = 2$, $t \in \{4, 5, 6\}$. Then the corresponding game Γ has values $s = t' = 2$. Thus, $u_{-1} = 1/2$, $u_0 = 1$, $u_1 = 4$, $u_2 = 14$, $u_3 = 50$, $\ldots$. The representations $R(N)$ over $\mathcal{U}$ of the first few positive integers N appear in Table 3. Consider the position $(4, 17) \in \Gamma_3$. By Corollary 5, we check $(\lfloor 4/3 \rfloor, \lfloor 17/3 \rfloor) = (1, 5)$ and their representations $(1, 11)$. Since 11 is not a left shift of 1 (but 1 ends in an even number of 0s), $(1, 5)$ is an N-position in Γ, hence $(4, 17)$ is an N-position in Γ_3. Now consider $(11, 37) \in \Gamma_3$, so $(\lfloor 11/3 \rfloor, \lfloor 37/3 \rfloor) = (3, 12)$, with representations $(3, 30)$. Since 3 ends in an even number of 0s and 30 is a left shift of 3, $(3, 30)$ is a P-position in Γ, hence $(11, 37)$ is a P-position in Γ_3.

Theorem 6. *Let $K, s, t \in \mathbb{Z}^+$. For Γ_K in misère play, $\mathscr{P} = \bigcup_{n=0}^{\infty} \{(E_n + \alpha, H_n + \beta) \mid \alpha, \beta \in \Omega_K\}$, where E_n and H_n are determined by two cases:*

(A) If $s > 1$ or $t > K$, then for $n \in \mathbb{Z}^0$,

$$\begin{cases} E_n = \text{mex}\{\{E_i \mid 0 \le i < n\} + \alpha, \{H_i \mid 0 \le i < n\} + \beta\}, \\ H_n = sE_n + \lceil t/K \rceil Kn + K. \end{cases} \tag{5}$$

(B) If $s = 1$ and $t \le K$, then $E_0 = H_0 = 2K$ and for $n \in \mathbb{Z}^+$,

$$\begin{cases} E_n = \text{mex}\{\{E_i \mid 0 \le i < n\} + \alpha, \{H_i \mid 0 \le i < n\} + \beta\}, \\ H_n = E_n + Kn. \end{cases} \tag{6}$$

Example 4. For $K = 3$, $s = 2$, $t \in \{4, 5, 6\}$, we display the first few P-generators of Γ_K in Table 4, which shows us how to determine $\mathscr{P}$ by using Eq. (5).

Proof. Let $E = \bigcup_{n=0}^{\infty} \{E_n\} + \alpha$ and $H = \bigcup_{n=0}^{\infty} \{H_n\} + \beta$ with $\alpha, \beta \in \Omega_K$. We firstly claim the following facts:

Fact A Suppose $s > 1$ or $t > K$.

I. Similar to Lemma 2(a) and (b), $E_n, H_n \in \mathcal{M}_K$ and it is easy to see that both E_n and H_n are strictly increasing sequences, for $n \in \mathbb{Z}^0$.

Table 3: Representations $R(N)$ over $\mathcal{U}$.

14	4	1	N
		1	1
		2	2
		3	3
	1	0	4
	1	1	5
	1	2	6
	1	3	7
	2	0	8
	2	1	9
	2	2	10
	2	3	11
	3	0	12
	3	1	13
1	0	0	14
1	0	1	15
1	0	2	16
1	0	3	17
1	1	0	18
1	1	1	19
1	1	2	20

Table 4: The first few P-generators of Γ_K for $K = 3$, $s = 2$, $t \in \{4, 5, 6\}$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
E_n	0	6	9	12	15	18	24	27	30	36	39	42	48
H_n	3	21	33	45	57	69	87	99	111	129	141	153	171

II. $E \cup H = \mathbb{Z}^0$ and $E \cap H = \emptyset$. In fact, $E \cup H = \mathbb{Z}^0$ follows from the definition of mex. Now suppose $E \cap H \neq \emptyset$. It follows Fact A.I that $E_m + \alpha' \neq H_n$ and $E_m \neq H_n + \beta'$ with $\alpha' > 0$, $\beta' > 0$, thus the only possibility is $E_m = H_n$ for two integers $m, n \in \mathbb{Z}^+$. If $m > n$ then $E_m = \text{mex}\{E_i + \alpha, H_i + \beta \mid 0 \leq i < m,\ \alpha, \beta \in \Omega_K\}$, which contradicts $E_m = H_n$; if $m \leq n$ then$H_n \geq sE_m + \lceil t/K \rceil Kn + K > E_m$, also contradicting $E_m = H_n$.

Fact B Suppose $s = 1$ and $t \leq K$.

I. $E_n, H_n \in \mathcal{M}_K$ for $n \in \mathbb{Z}^0$ and E_n, H_n are strictly increasing sequences for $n \in \mathbb{Z}^+$.

II. $E \cup H = \mathbb{Z}^0$ and $E \cap H = \{2K\}$. Its proof is similar to that of Fact A.II.

Proof of Fact I. Let (x, y) with $x \leq y$ be a position in $\mathscr{P}$. There exist some $n \in \mathbb{Z}^0$ and $\alpha, \beta \in \Omega_K$ such that $(x, y) = (E_n + \alpha, H_n + \beta)$.

It is easy to check that no move of type I from (x, y) can terminate in $\mathscr{P}$. Then suppose $(x, y) \to (x', y') \in \mathscr{P}$ by a type II move, and there exists some m such that $(x', y') = (E_m + \alpha, H_m + \beta)$. Thus for both cases (A) and (B), we have $k = E_n - E_m > 0$, $\ell = H_n - H_m$ and $0 < k \leq \ell = s(E_n - E_m) + \lceil t/K \rceil K(n - m) \geq sk + t$, which contradicts Eq. (1).

Proof of Fact II. Let (x, y) with $x \leq y$ be a position not in $\mathscr{P}$. By Facts A.II and B.II, we have either $x = H_n + \beta$ or $x = E_n + \alpha$, for some $n \in \mathbb{Z}^0$ and $\alpha, \beta \in \Omega_K$.

Case (i) $x = H_n + \beta$. Now $y \geq E_n + K$. Let $y = qK + \alpha$, $q \in \mathbb{Z}^0$, and $\alpha \in \Omega_K$. Then move $y \to E_n + \alpha$, since $0 < y - E_n - \alpha \in \mathcal{M}_K$.

Case (ii) $x = E_n + \alpha$. In this case, we have $y > H_n + K - 1$ or $x \leq y < H_n$. Let $y = qK + \beta$, where $q \in \mathbb{Z}^0$, and $\beta \in \Omega_K$. If $y > H_n + K - 1$, then move $y \to H_n + \beta$, since $0 < y - H_n - \beta \in \mathcal{M}_K$. If $x \leq y < H_n$, we consider two subcases: (ii-A) $s > 1$ or $t > K$; (ii-B) $s = 1$ and $t \leq K$.

(ii-A) $s > 1$ or $t > K$.

For $n = 0$, we have $x \leq y < K = H_0$, the next player wins without doing anything.

For $n \geq 1$. If $x \leq y < sE_n + \lceil t/K \rceil K + K$, move $(x, y) \to (\alpha, K + \beta)$. This is a legal move, since $k = E_n$, $\ell = y - K - \beta$, and $0 \leq \ell - k < sE_n + \lceil t/K \rceil K - \beta - E_n \leq (s-1)E_n + \lceil t/K \rceil K - K < (s-1)k + t$. If $sE_n + \lceil t/K \rceil K + K \leq y < H_n$, put $m = \lfloor (y - sE_n - K - \beta)/(\lceil t/K \rceil K) \rfloor$ and move $(x, y) \to (E_m + \alpha, H_m + \beta)$. This move is legal:

(a) $0 < k \in \mathcal{M}_K$. Clearly $k = E_n - E_m \in \mathcal{M}_K$. It suffices to prove that $0 \leq m < n$. Note that $y - sE_n - K \geq \lceil t/K \rceil K \geq K > \beta$, so $(y - sE_n - K - \beta)/(\lceil t/K \rceil K) > 0$, thus $m = \lfloor (y - sE_n - K - \beta)/(\lceil t/K \rceil K) \rfloor \geq 0$. On the other hand, $y - sE_n - K - \beta < H_n - sE_n - K = \lceil t/K \rceil Kn$, and so $m = \lfloor (y - sE_n - K - \beta)/(\lceil t/K \rceil K) \rfloor \leq (y - sE_n - K - \beta)/(\lceil t/K \rceil K) < n$.

(b) $0 < \ell \in \mathcal{M}_K$. It is obvious that $\ell = y - H_m - \beta = qK - H_m \in \mathcal{M}_K$. Now $m \leq (y - sE_n - K - \beta)/(\lceil t/K \rceil K)$, So $y \geq \lceil t/K \rceil Km + sE_n + K + \beta = H_m + \beta + s(E_n - E_m) > H_m + \beta$.

(c) $k \leq \ell < sk + t$. By above, $m > (y - sE_n - K - \beta)/(\lceil t/K \rceil K) - 1$, i.e., $y < \lceil t/K \rceil K(m + 1) + sE_n + K + \beta$. So $y - H_m - \beta < s(E_n - E_m) + \lceil t/K \rceil K$, thus we have $\ell = y - H_m - \beta \leq s(E_n - E_m) + \lceil t/K \rceil K - K < s(E_n - E_m) + t = sk + t$. On the other hand, by (b), $\ell = y - H_m - \beta \geq s(E_n - E_m) \geq E_n - E_m = k$.

(ii-B) $s = 1$ and $t \le K$.

If $n = 0$, then $2K + \alpha = x \le y < H_0 = 2K$ is impossible; if $n = 1$ then $0 \le x \le y \le K - 1$, thus the next player wins without doing anything. It remains to consider the case $n \ge 2$:

Put $m = \lfloor (y - E_n - \beta)/K \rfloor$ and move $(x, y) \to (E_m + \alpha, H_m + \beta)$. This move is legal:

(a) $0 < k = E_n - E_m \in \mathcal{M}_K$. As above, we only need to prove that $0 \le m < n$. since $y \ge E_n + \beta$, then $m = \lfloor (y - E_n - \beta)/K \rfloor \ge 0$. On the other hand, $y - E_n - \beta < H_n - E_n = Kn$, and so $m = \lfloor (y - E_n - \beta)/K \rfloor \le (y - E_n - \beta)/K < n$.

(b) $0 < \ell \in \mathcal{M}_K$. Obviously, $\ell = y - H_m - \beta = qK - H_m \in \mathcal{M}_K$. Now $m \le (y - E_n - \beta)/K$. Thus we have $y \ge Km + E_n + \beta = H_m + \beta + E_n - E_m > H_m + \beta$.

(c) $k \le \ell < k + t$. On the one hand, $m > (y - E_n - \beta)/K - 1$, i.e., $y < K(m + 1) + E_n + \beta$. Thus $\ell = y - H_m - \beta < K(m+1) + E_n - E_m - Km = E_n - E_m + K$. Note that both $y - H_m - \beta$ and $E_n - E_m + K$ are in $\mathcal{M}_K$, so $\ell = y - H_m - \beta \le E_n - E_m < k + t$. On the other hand, by (b), $\ell = y - H_m - \beta \ge E_n - E_m = k$. ■

Theorem 6 provides a recursive winning strategy for Γ_K in misère play, which is exponential. We now examine whether Γ_K has a poly-time winning strategy or not.

In Section 7 of [9], three characterizations, recursive, algebraic and arithmetic, are given for the P-positions of Generalized Wythoff in misère play, which is the case $K = s = 1$ of Γ_K. Take the recursive and algebraic characterizations for example, denote by $\{(E'_n, H'_n)\}_{n \ge 0}$ the P-positions of Generalized Wythoff with parameter $t \in \mathbb{Z}^+$, we have

(i) Recursive characterization

For $t = 1$: $(E'_0, H'_0) = (2, 2)$, $E'_n = \text{mex}\{E'_i, H'_i \mid 0 \le i < n\}$, $H'_n = E'_n + n$ $(n \ge 1)$.

For $t > 1$: $E'_n = \text{mex}\{E'_i, H'_i \mid 0 \le i < n\}$, $H'_n = E'_n + tn + 1$ $(n \ge 0)$.

(ii) Algebraic characterization

For $t = 1$: $(E'_0, H'_0) = (2, 2)$, $(E'_1, H'_1) = (0, 1)$,

$E'_n = \lfloor n\phi \rfloor$, $H'_n = \lfloor n\phi^2 \rfloor$ $(n \ge 2)$, where $\phi = (1 + \sqrt{5})/2$.

For $t > 1$: $E'_n = \lfloor n\alpha + \gamma \rfloor$, $H'_n = \lfloor n\beta + \delta \rfloor$ $(n \ge 0)$, where $\alpha = (2 - t + \sqrt{t^2 + 4})/2$, $\beta = \alpha + t$, $\gamma = 1/\alpha$, $\delta = \gamma + 1$.

For the arithmetic winning strategy, which involves a continued fraction and two numeration systems, p-system and q-system, we refer the reader to Section 7 of [9]. It was pointed out there that the first one strategy is exponential while the last two provide poly-time strategies for Generalized Wythoff. Now there exists a connection between Γ_K with parameters $K, s, t \in \mathbb{Z}^+$ and Generalized Wythoff but with parameter $t' = \lceil t/K \rceil$, t, K as in Γ_K.

Theorem 7. *Let $s = 1$. $E'_n = E_n/K$, $H'_n = H_n/K$, where $\{(E_n, H_n)\}_{n \ge 0}$ and $\{(E'_n, H'_n)\}_{n \ge 0}$ the P-generators of Γ_K and the P-positions of Generalized Wythoff.*

Proof. This follows by the same method as in the proof of Lemma 4. ■

Corollary 8. *In misère play, (x, y) is a P-position of Γ_K $(s = 1)$ if and only if $(\lfloor x/K \rfloor, \lfloor y/K \rfloor)$ is a P-position of Generalized Wythoff.*

Proof. Directly follows from Lemma 7. $\blacksquare$

Now based on this simple connection, together with the poly-time winning strategy for Generalized Wythoff, Γ_K has a poly-time winning strategy for $s = 1$. However, for $s > 1$, there is no poly-time winning strategy yet.

4 Rook type restrictions of (s, t)-Wythoff's game

In this section, let $\mathbb{Z}^{even} = \{2n \mid n \in \mathbb{Z}^0\}$, $\mathbb{Z}^{odd} = \{2n + 1 \mid n \in \mathbb{Z}^0\}$. We also need the function δ_n for $n \in \mathbb{Z}^0$, which has been defined in [12]:

$$\delta_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

4.1 The P-positions of Γ_{OA}

In Γ_{OA}, asymmetry of the game rules implies that (a, b) is not necessarily identical to (b, a).

Theorem 9. *Let $s, t \in \mathbb{Z}^+$. For Γ_{OA} in normal play, then*

(1) If $s = t = 1$, $\mathscr{P} = \bigcup_{n=0}^{\infty} \{(2n, 0), (2n + 1, 2)\}$.

(2) If $s + t > 2$, $\mathscr{P} = \bigcup_{n=0}^{\infty} \{(A_n, B_n)\}$, where $A_n = n$, $B_n = \delta_n(sn + (n + 1)t/2)$.

Proof. (1) Clearly for stability property of $\mathscr{P}$. Suppose (a, b) is a position not in $\mathscr{P}$. If $a = 2n$ for some $n \in \mathbb{Z}^0$, move $(a, b) \to (2n, 0)$. If $a = 2n + 1$ for some $n \in \mathbb{Z}^0$, then $b \in \{0, 1\}$ or $b \geq 3$. For the former, we move $(a, b) \to (2n, 0)$. Otherwise, move $(a, b) \to (2n + 1, 2)$.

(2) **Proof of Fact I**. Given $(A_n, B_n) \in \mathscr{P}$. Suppose that $(A_n, B_n) \to (A_m, B_m) \in \mathscr{P}$. Then $n \in \mathbb{Z}^{even}$ cannot happen, since $B_n = 0 < B_m$. Thus we have $n \in \mathbb{Z}^{odd}$. If m is also odd, then $k = n - m \geq 2$, thus $\ell = B_n - B_m = s(n - m) + (n - m)t/2 \geq sk + t$, which contradicts the condition $0 < k \leq \ell < sk + t$. But if m is even, then we have $k = n - m > 0$ and $\ell = B_n = sn + (n + 1)t/2 \geq sk + t$, another contradiction.

Proof of Fact II. Let (x, y) be a position not in $\mathscr{P}_1$. If x is even, then move $y \to 0$. If x is odd, there exists some n such that $x = A_n = n$ and we have either $y > B_n$ or $0 \leq y < B_n$. If $y > B_n$, then move $y \to B_n$. If $0 \leq y < B_n$ we distinguish the following four cases:

- $y = 0$. Then move $(x, y) \to (x - 1, 0)$.
- $1 \leq y < x$. We move $(x, y) \to (x - y - \delta_y + 1, 0) \in \mathscr{P}$ on account of $x - y - \delta_y + 1 \in \mathbb{Z}^{even}$.

This move is legal, since $k = y + \delta_y - 1 > 0$, $\ell = y > 0$, and $0 \leq \ell - k \leq 1 < (s - 1)k + t$.

• $x \le y < sx + t$. Then move $(x, y) \to (0, 0)$, which satisfies the condition Eq. (1) with $k = A_n$, $\ell = y$.

• $sx + t \le y < B_n$. Put $m = 2\lfloor (y - sx - t)/t \rfloor + 1$ and move $(x, y) \to (A_m, B_m)$. This move is legal, since (a) $m < n$, (b) $y > B_m$, (c) $A_n - A_m \le y - B_m < s(A_n - A_m) + t$. Indeed,

(a) $y - sx - t < B_n - sx - t = (n-1)t/2$, so $m \le 2(y - sx - t)/t + 1 < n$;

(b) $m \le 2(y - sx - t)/t + 1$, so $y \ge (m-1)t/2 + sx + t = B_m + s(n-m) > B_m$;

(c) $m > 2((y-sx-t)/t-1)+1 = 2(y-sx-t)/t-1$, so $y < (m+1)t/2+sx+t = sn+(m+3)t/2$; by (b), $y - B_m \ge n - m = A_n - A_m$, hence,

$$A_n - A_m \le y - B_m < sn + (m+3)t/2 - sm - (m+1)t/2 = s(A_n - A_m) + t.$$

Thus Eq. (1) is satisfied. ■

Theorem 10. *Put $s, t \in \mathbb{Z}^+$. For Γ_{OA} in misère play, then*

(1) If $s = t = 1$, $\mathscr{P} = (0, 1) \cup \bigcup_{n=0}^{\infty} \{(2n+1, 0), (2n+2, 2)\}$.

(2) If $s + t > 2$, $\mathscr{P} = \bigcup_{n=0}^{\infty} \{(E_n, H_n)\}$, where $E_n = n$, $H_n = (1 - \delta_n)(sn + tn/2 + 1)$.

Proof. (1) Both stability and absorbing properties of $\mathscr{P}$ when $s = t = 1$ are simple. The details are left to the reader.

(2) **Proof of Fact I**. Suppose a move from (E_n, H_n) produces another position of the form (E_m, H_m). It is easy to see that the only possibility is that n is even. If m is also even, this implies $k = n - m \ge 2$, then $\ell = H_n - H_m = s(n-m) + t(n-m)/2 \ge sk + t$, which contradicts Eq. (1). If n is even but m is odd, then $k = n - m > 0$, thus $\ell = H_n - H_m = sn + tn/2 + 1 \ge s(n-m) + tn/2 \ge sk + t$, another contradiction.

Proof of Fact II. Let (x, y) be a position not in $\mathscr{P}$. We will show that there exists a legal move such that $(x, y) \to (E_n, H_n)$. Put $x = E_n = n$ for some $n \in \mathbb{Z}^+$. If $x = 0$, then $(E_0, H_0) = (0, 1)$. For $(0, 0)$, the next player wins without doing anything; for $y > 1$, we only need to move $y \to 1$. If x is odd, then move $y \to 0 = H_n$. If x is even, this implies $y > H_n$ or $0 \le y < H_n$. For the former, we move $y \to H_n$; while for the latter, we distinguish the following four cases:

• $y = 0$. Then move $(x, y) \to (E_n - 1, 0) \in \mathscr{P}$.

• $1 \le y \le x$. In this case, move $(x, y) \to (x - y - \delta_y + 1, 0) \in \mathscr{P}$, since $x - y - \delta_y + 1 > 0$ is odd. This move is legal: (a) $k = y - 1 + \delta_y > 0$, (b) $\ell = y > 0$, (c) $0 \le \ell - k \le 1 < (s-1)k + t$.

• $x < y < sx + t + 1$. we move $(x, y) \to (E_0, H_0) = (0, 1)$, which satisfies Eq. (1) with $k = x$, $\ell = y - 1 < sx + t$.

• $sx + t + 1 \le y < H_n$. Put $m = 2\lfloor (y - sx - 1)/t \rfloor$ and move $(x, y) \to (E_m, H_m)$. This is a legal move, since (a) $m < n$, (b) $y > H_m$, and (c) $E_n - E_m \le y - H_m < s(E_n - E_m) + t$. Indeed,

(a) $y - sx - 1 < H_n - sn - 1 = nt/2$, so $m = 2\lfloor (y - sx - 1)/t \rfloor \le 2(y - sx - 1)/t < n$;

(b) $m \le 2(y - sx - 1)/t$, so $y \ge mt/2 + sx + 1 = H_m + s(n-m) > H_m$;

(c) $m > 2(y-sx-1)/t-2$, thus $y < sn+(m+2)t/2+1$; by (b), $y-H_m \geq n-m = E_n-E_m$. Therefore, $E_n - E_m \leq y - H_m < sn + mt/2 + t + 1 - sm - mt/2 - 1 = s(E_n - E_m) + t$, thus Eq. (1) is satisfied. ■

4.2 The P-positions of Γ_{OO}

Obviously, the game rules of Γ_{OO} is symmetrical, so we say (a, b) is a P-position, meaning that (b, a) is also a P-position.

Theorem 11. *Given $s, t \in \mathbb{Z}^+$. For Γ_{OO} in normal play, $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(0, 2n)\}$.*

Proof. A move from $(0, 2n)$ clearly leads to a position not in $\mathscr{P}$. Let (x, y) with $x \leq y$ be a position not in $\mathscr{P}$. If $x = 0$ and y is odd, only move $y \to y - 1$. Consider $x > 0$. If $x, y \in \mathbb{Z}^{odd}$ or $x, y \in \mathbb{Z}^{even}$, then move $(x, y) \to (0, y - x) \in \mathscr{P}$. Otherwise, we take the entire pile with an odd number of tokens. ■

Theorem 12. *Given $s, t \in \mathbb{Z}^+$. For Γ_{OO} in misère play,*

$$\mathscr{P} = \begin{cases} \{(0, 2n+1), (2, 2n) \mid n \in \mathbb{Z}^+\}, & \text{if } s = t = 1, \\ \{(0, 2n+1) \mid n \in \mathbb{Z}^+\}, & \text{if } s + t > 2. \end{cases}$$

Proof. Stability property of $\mathscr{P}$ is straightforward. Let (x, y) with $x \leq y$ be a position not in $\mathscr{P}$. It suffices to show that from (x, y) there is a move terminating in $\mathscr{P}$. Consider three cases:

- $x = 0$. Clearly for $y = 0$. If $y > 0$, then y is even and move $y \to y - 1$.
- $x = 1$. Then move $(1, y) \to (0, y - 1 + \delta_y) \in \mathscr{P}$, since $y - 1 + \delta_y$ is odd.
- $x \geq 2$. For $s = t = 1$. If $x = 2$, then move $(x, y) \to (2, y - 1)$; if $x \geq 3$, we move $(x, y) \to (2 - 2\delta_{y-x},\ y - x - 2\delta_{y-x} + 2)$ by taking $x + 2\delta_{y-x} - 2 > 0$ tokens from both piles. Note that if $y - x$ is odd, we have $(2 - 2\delta_{y-x},\ y - x - 2\delta_{y-x} + 2) = (0, y - x) \in \mathscr{P}$; if $y - x$ is even, then $(2 - 2\delta_{y-x},\ y - x - 2\delta_{y-x} + 2) = (2, y - x + 2) \in \mathscr{P}$.

For $s + t > 2$, we move $(x, y) \to (0, y - x - \delta_{y-x} + 1) \in \mathscr{P}$, since $y - x - \delta_{y-x} + 1$ is odd. This is a legal move, since: (a) $k = x - 1 + \delta_{y-x} > 0$, (b) $\ell = x$, and (c) $0 \leq \ell - k = 1 - \delta_{y-x} \leq 1 < s + t - 1 \leq (s - 1)k + t$. ■

4.3 The P-positions of Γ_{OE}

In Γ_{OE}, (a, b) is not necessarily identical to (b, a) because of asymmetry.

Theorem 13. *Let $s = t = 1$. For Γ_{OE} in normal play,*

$$\mathscr{P} = \bigcup_{n=0}^{\infty}\{(2n, 0), (2n, 1), (2n + 1, 4n + 3), (2n + 1, 4n + 4)\}.$$

Proof. The proof of stability property of $\mathscr{P}$ is simple, we leave the details to the reader. Now we prove absorbing property of $\mathscr{P}$. Let (x, y) be a position not in $\mathscr{P}$.

If x is even, then move $(x, y) \to (x, \delta_y)$.

If $x = 2n+1$ for $n \in \mathbb{Z}^0$. Then $y \geq 4n+5$ or $0 \leq y \leq 4n+2$. For the former, we move $(x, y) \to (2n+1, 4n+4-\delta_y)$. For the latter, if $y = 0$, then move $(x, y) \to (2n, 0)$; if $1 \leq y \leq x+1$, then move $(x, y) \to (x-y-\delta_y+1, 1-\delta_y)$ by taking $y+\delta_y-1 > 0$ tokens from both piles. Since $x-y-\delta_y+1$ is even and $1-\delta_y \in \{0, 1\}$, thus $(x-y-\delta_y+1, 1-\delta_y) \in \mathscr{P}$. Finally, if $x+2 \leq y \leq 4n+2$. Then we move $(x, y) \to (y+\delta_y-x-2, 2y+\delta_y-2x-2)$ by taking $2x-y-\delta_y+2$ $(\geq 2-\delta_y > 0)$ tokens from both piles. The proof is completed by showing that $(y+\delta_y-x-2, 2y+\delta_y-2x-2) \in \mathscr{P}$:

Let $y+\delta_y-x-2 = \phi$. Then $2y+\delta_y-2x-2 = 2\phi+2-\delta_y \in \{2\phi+1, 2\phi+2\}$. Since $y+\delta_y$ is even, x is odd, we get ϕ is odd. It is easy to see that $(\phi, 2\phi+1), (\phi, 2\phi+2) \in \mathscr{P}$. ■

Theorem 14. *Let $s+t > 2$. For Γ_{OE} in normal play, $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(A_n, B_n), (A_n, B'_n)\}$, where for $n \geq 0$,*

$$\begin{cases} A_n = n, \\ B_n = (s+t+1)A_n + 1, \\ B'_n = \delta_n(B_n - 1). \end{cases}$$

Example 5. For $s = 2$, $t = 3$, we display the first few P-positions of Γ_{OE} in Table 5.

Table 5: The first few P-positions of Γ_{OE} for $s = 2$, $t = 3$ in normal play.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	1	7	13	19	25	31	37	43	49	55	61	67	73	79	85
B'_n	0	6	0	18	0	30	0	42	0	50	0	66	0	78	0

Proof. **Proof of Fact I**. Given $(A_n, B_n) \in \mathscr{P}$. Suppose that $(A_n, B_n) \to (A_m, B_m) \in \mathscr{P}$. Then we have $k = n-m > 0$, and $\ell = (s+t+1)(n-m) > sk+t$, which contradicts Eq. (1).

Suppose that $(A_n, B_n) \to (A_m, B'_m) \in \mathscr{P}$. In this case, we have $k = n-m > 0$, and

$$\begin{aligned} \ell = B_n - B'_m \quad &= \begin{cases} (s+t+1)(n-m)+1 & \text{if } m \in \mathbb{Z}^{odd} \\ (s+t+1)n+1 & \text{if } m \in \mathbb{Z}^{even} \end{cases} \\ &> (s+t+1)k > sk+t, \end{aligned}$$

also contradicting Eq. (1).

Given $(A_n, B'_n) \in \mathscr{P}$. Notice that if n is even and so $B'_n = 0$, then any move from $(A_n, 0)$ cannot lead to a position in $\mathscr{P}$. Now suppose n is odd and so $B'_n = B_n - 1$. If $(A_n, B'_n) \to (A_m, B_m) \in \mathscr{P}$, then we have $k = n - m > 0$, and $\ell = B_n - B_m - 1 = (s+t+1)(n-m) - 1 = (s+t+1)k - 1 \geq sk + t$, a contradiction; if $(A_n, B'_n) \to (A_m, B'_m) \in \mathscr{P}$, then we get $k = n - m > 0$, and

$$\ell = B_n - 1 - B'_m = \begin{cases} (s+t+1)(n-m) & \text{if } m \in \mathbb{Z}^{odd} \\ (s+t+1)n & \text{if } m \in \mathbb{Z}^{even} \end{cases} \geq (s+t+1)k > sk + t,$$

another contradiction.

Proof of Fact II. Let (x, y) be a position not in $\mathscr{P}$. We show that there exists a legal move such that $(x, y) \to (A_n, B_n)$ or (A_n, B'_n).

Put $x = A_n$ for some $n \in \mathbb{Z}^0$. We distinguish two cases: (i) x is even; (ii) x is odd.

Case (i) $x = A_n = n$ is even.

In this case, note first that $B_n = (s+t+1)n + 1$ is odd and $B'_n = 0$. The fact $(x, y) \notin \mathscr{P}$ implies that $y > B_n$ or $0 < y < B_n$. For $y > B_n$, if y is even, then move $y \to B'_n$; if y is odd, we move $y \to B_n$. For $0 < y < B_n$, we proceed by distinguishing three subcases:

- $1 \leq y < x$. Then move $(x, y) \to (x - y - \delta_y, 0) \in \mathscr{P}$. This move is legal, since (a) $k = y \geq 1$, (b) $\ell = y + \delta_y \geq 2$, (c) $0 \leq \ell - k = \delta_y \leq 1 < (s-1)k + t$.
- $x \leq y \leq B_n - 2$. In this subcase, put $m = \lfloor (y - x)/(s+t) \rfloor$ and move $(x, y) \to (A_m, B_m)$. This move is legal:

(a) $0 \leq m < n$. Indeed, $0 \leq y - x \leq B_n - x - 2 = (s+t)n - 1 < (s+t)n$, so $0 \leq m = \lfloor (y-x)/(s+t) \rfloor \leq (y-x)/(s+t) < n$.

(b) By the definition of m, we have $(y-x)/(s+t) - 1 < m \leq (y-x)/(s+t)$, i.e.,

$$(s+t)m \leq y - x < (s+t)(m+1). \tag{7}$$

Thus, $y \geq (s+t)m + x = B_m + (A_n - A_m) - 1 \geq B_m$ by virtue of $A_n - A_m \geq 1$.

If $y = B_m$ then $A_n - A_m = 1$. This is a legal move only from the first pile.

If $y - B_m \geq 1$, then it follows from Eq. (7) that $|(y - B_m) - (x - A_m)| = |y - x - (s+t)m - 1| < s + t - 1 \leq (s-1)\lambda + t$, where $\lambda := \{A_n - A_m, y - B_m\} \geq 1$.

- $y = B_n - 1$. Then move $y \to 0$ by taking $y \in \mathbb{Z}^{even}$ tokens from the second pile.

Case (ii) $x = A_n = n$ is odd.

In this case, $B'_n = B_n - 1$, then $y > B_n$ or $0 \leq y \leq B_n - 2$. For $y > B_n$. we see $B_n = (s+t+1)n + 1$ is odd if $s+t$ is odd, or B_n is even if $s+t$ is even. Thus if $(y, s+t \in \mathbb{Z}^{odd})$ or $(y, s+t \in \mathbb{Z}^{even})$, then we move $y \to B_n$; if $(y \in \mathbb{Z}^{even}, s+t \in \mathbb{Z}^{odd})$ or $(y \in \mathbb{Z}^{odd}, s+t \in \mathbb{Z}^{even})$, we move $y \to B'_n$. For $0 \leq y \leq B_n - 2$. We consider the following three subcases:

- $y = 0$. We just move $(x, y) \to (A_n - 1, 0) \in \mathscr{P}$.

• $1 \le y < x$. In this subcase, we move $(x,y) \to (x-y-1+\delta_y, 0) \in \mathscr{P}$. This is a legal move, since (a) $k = y > 0$, (b) $\ell = y + 1 - \delta_y > 0$, (c) $0 \le \ell - k = \delta_y \le 1 < (s-1)k + t$.

• $x \le y \le B_n - 2$. We move $(x,y) \to (A_m, B_m)$ with $m = \lfloor (y-x)/(s+t) \rfloor$. This follows from the same method as in case (i). ■

Theorem 15. *Let $s = t = 1$. For Γ_{OE} in misère play,*

$$\begin{aligned} \mathscr{P} &= \{(0,2),(0,3),(2,3),(2,6)\} \\ &\quad \cup \bigcup_{n=0}^{\infty} \left\{ \begin{array}{l} (2n+1,0),(2n+1,1), \\ (2n+4,4n+9),(2n+4,4n+10) \end{array} \right\}. \end{aligned}$$

Proof. Stability property of $\mathscr{P}$ is simple. We are left with the task of proving absorbing property of $\mathscr{P}$. It is easy to check that $(0,2),(0,3),(2,3),(2,6)$ are all P-positions by the knowledge of Example 1 in Section 2. If $x = 2n+1$ for some $n \in \mathbb{Z}^0$, then move $(x,y) \to (x, \delta_y)$. If $x = 2n+4$ for some $n \in \mathbb{Z}^0$, then $y > 4n+10$ or $0 \le y < 4n+9$. For the former, we move $(x,y) \to (2n+4, 4n+9)$ (if y is odd) or $(x,y) \to (2n+4, 4n+10)$ (if y is even). For the latter, we consider three cases:

• $0 \le y \le x$. If $y = 0$, then move $x \to 2n+1$, or else, we move $(x,y) \to (x-y-\delta_y+1, 0) \in \mathscr{P}$, which satisfies Eq. (1) with $k = y + \delta_y - 1$ and $\ell = y$.

• $x < y \le x+4$. If $y \in \{x+1, x+4\}$, then remove $x-2$ tokens from both piles leading to $(2,3)$ or $(2,6)$; if $y \in \{x+2, x+3\}$, remove x tokens from both piles leading to $(0,2)$ or $(0,3)$.

• $x+5 \le y < 4n+9$. Then move $(x,y) \to (y-x-2+\delta_y, 2y-2x-2+\delta_y)$ by taking $2x-y+2-\delta_y$ tokens from both piles. Let $y-x-2+\delta_y = \phi$. Clearly ϕ is even and because of $\phi \ge 3+\delta_y$, then $\phi \ge 4$. Thus there exists some $n \in \mathbb{Z}^0$ such that $\phi = 2n+4$. Furthermore, $2y-2x-2+\delta_y = 2\phi+2-\delta_y \in \{2\phi+1, 2\phi+2\} = \{4n+9, 4n+10\}$. Hence, $(y-x-2+\delta_y, 2y-2x-2+\delta_y) \in \mathscr{P}$. ■

Theorem 16. *Let $s+t > 2$. For Γ_{OE} in misère play, $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(E_n,H_n),(E_n,H'_n)\}$, where for $n \in \{0,1,2\}$,*

n	0	1	2
E_n	0	1	2
H_n	2	0	2s+t+3
H'_n	3	1	2s+t+4

and for $n \geq 3$,

$$\begin{cases} E_n = n, \\ H_n = (s+t+1)E_n + 2 - \delta_s - t, \\ H'_n = (1-\delta_n)(H_n - 1). \end{cases} \tag{8}$$

Example 6. For $s = t = 3$, we display the first few P-positions of Γ_{OE} in Table 6.

Table 6: The first few P-positions of Γ_{OE} for $s = t = 3$ in misère play.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
H_n	2	0	9	15	20	25	30	35	40	45	50	55	60	65	70
H'_n	3	1	10	0	19	0	29	0	39	0	49	0	59	0	64

Proof. The proof is tedious. We first prove stability property of $\mathscr{P}$. Given a position (E_n, H_n) (or (E_n, H'_n)) in $\mathscr{P}$, if $n < 3$, we leave it to the reader to verify that a legal move from (E_n, H_n) (or (E_n, H'_n)) leads to a position not in $\mathscr{P}$. For $n \geq 3$, it is easy to check that a legal move from (E_n, H_n) (or (E_n, H'_n)) cannot land in $\bigcup_{i<3}\{(E_i, H_i), (E_i, H'_i)\}$. Let $m \geq 3$.

Now suppose $(E_n, H_n) \to (E_m, H_m)$, then $k = n - m > 0$ and $\ell = H_n - H_m = (s+t+1)(n-m) > sk + t$, which contradicts Eq. (1).

Suppose that $(E_n, H_n) \to (E_m, H'_m)$, then $k = n - m > 0$. And if m is odd, $\ell = H_n - 0 = (s+t+1)n + 2 - \delta_s - t > (n-m)s + (n-1)t > sk + t$; if m is even, then $\ell = H_n - (H_m - 1) = (s+t+1)(n-m) + 1 > sk + t$. Both contradict Eq. (1).

Next suppose $(E_n, H'_n) \to (E_m, H_m)$. It is impossible that n is odd. Indeed, if so, it follows by the definition of δ_n that $H'_n = 0 < H_m$. If n is even, then we have $H'_n = H_n - 1$ and $k = n - m > 0$, but $\ell = (H_n - 1) - H_m = (s+t+1)(n-m) - 1 \geq sk + t$, another contradiction.

Finally, suppose that $(E_n, H'_n) \to (E_m, H'_m)$. As above, n is even. If m is also even, then $\ell = (H_n - 1) - (H_m - 1) = (s+t+1)(n-m) > sk + t$; if n is even but m is odd, then $\ell = H_n - 1 = (s+t+1)n + 1 - \delta_s - t > (n-m)s + (n-1)t > sk + t$. In a word, this move is also illegal.

We next prove absorbing property of $\mathscr{P}$. Let (x, y) be a position not in $\mathscr{P}$. Put $x = E_n = n$ for some $n \in \mathbb{Z}^0$.

If $x = 0$, then $y \in \{0, 1\}$ or $y \geq 4$. Obviously, $(0, 0)$ and $(0, 1)$ are N-positions. For $y \geq 4$, then move $(0, y) \to (0, 2 + \delta_y) \in \mathscr{P}$.

If $x = 1$, then $y \geq 2$, move $(1, y) \to (1, \delta_y) \in \mathscr{P}$.

If $x = 2$, we have either $y > 2s + t + 4$ or $0 \leq y < 2s + t + 3$.

Case (i) $y > 2s+t+4$. If ($y \in \mathbb{Z}^{odd}$ and $t \in \mathbb{Z}^{even}$) or ($y \in \mathbb{Z}^{even}$ and $t \in \mathbb{Z}^{odd}$), we move $(2,y) \to (2, 2s+t+3) \in \mathscr{P}$ since $\ell = y-2s-t-3 > 0$ is even; if ($y, t \in \mathbb{Z}^{odd}$) or ($y, t \in \mathbb{Z}^{even}$), then we move $(2,y) \to (2, 2s+t+4) \in \mathscr{P}$ because $y-2s-t-4$ is always even.

Case (ii) $0 \le y < 2s+t+3$. If $y=0$, we move $(2,0) \to (1,0)$; if $y \in \{1,2,3\}$, we move $(2,y) \to (1,1)$; if $y=4$, move $(2,4) \to (0,2)$, if $5 \le y < 2s+t+3$, then move $(2,y) \to (0,3)$, which satisfies Eq. (1) with $k=2$ and $\ell = y-3$.

If $x \ge 3$, we proceed by distinguish two cases: (i) x is odd; (ii) x is even.

Case (iii) $x = n$ is odd.

In this case, $H'_n = 0$ and so $y > H_n$ or $0 < y < H_n$. For $y > H_n$, if y is even, we move $y \to 0$; if y is odd, then move $y \to H_n$ as $\ell = y - H_n = y-(n-1)(s+t)-(s-\delta_s)-n+2$ is even. The case $0 < y < H_n$ is rebarbative. With patience we proceed by distinguishing seven subcases:

- $0 < y < x$. If y is even, we move $y \to H'_n = 0$; if y is odd, we move $(x,y) \to (x-y-1, 0)$ since $x-y-1$ is odd. Obviously this move satisfies Eq. (1) with $k=y$ and $\ell = y+1$.
- $y \in \{x, x+1\}$. Then move $(x,y) \to (1,1)$.
- $x+2 \le y \le x+2s+t$. We move $(x,y) \to (0,3)$, which is legal, since (a) $k = x > 0$, (b) $\ell = y-3 \ge x-1 > 0$, (c) $|\ell - k| \le 2s+t-3 < 2(s-1)+t \le (s-1)\lambda + t$, where $\lambda := \min\{x, y-3\} \ge 2$.
- $y = x+2s+t+1$. We move $(x,y) \to (2, 2s+t+3) \in \mathscr{P}$ by removing $x-2 > 0$ tokens from both piles.
- $x+2s+t+2 \le y \le x+3s+2t$. Then move $(x,y) \to (2, 2s+t+4) \in \mathscr{P}$. This move is legal, since (a) $k = x-2 > 0$, (b) $\ell = y-(2s+t+4) \ge x-2 > 0$, (c) $|\ell - k| \le s+t-2 < s+t-1 \le (s-1)k+t$.
- $x+3s+2t+1 \le y < H_n - 1$. Put $m = \lfloor (y-x+t-1+\delta_s)/(s+t) \rfloor$ and move $(x,y) \to (E_m, H_m)$. This move is also legal, since

(a) $n > m \ge 3$. Indeed, $y-x+t-1+\delta_s < H_n - x+t-2+\delta_s = (s+t)n$, thus we have $m \le (y-x+t-1+\delta_s)/(s+t) < n$. On the other hand, $y-x+t-1+\delta_s \ge x+3s+2t+1-(x-t+1-\delta_s) \ge 3(s+t)$. so $m \ge 3$ and $k = n-m > 0$.

(b) $y \ge H_m$. By the definition of m, $(y-x-s-1+\delta_s)/(s+t) < m \le (y-x+t-1+\delta_s)/(s+t)$, i.e.,

$$(s+t)m - t + 1 - \delta_s \le y - x < (s+t)m + s + 1 - \delta_s. \tag{9}$$

Thus $y \ge (s+t)m + x - t + 1 - \delta_s = H_m + (E_n - E_m) - 1 \ge H_m$ by virtue of $E_n - E_m \ge 1$.

If $y = H_m$, then $E_n - E_m = 1$. This is a legal move only from the first pile.

If $y - H_m \ge 1$, then it follows from Eq. (9) that $|(y-H_m)-(x-E_m)| = |y-x-(s+t)m-2+t+\delta_s| < s+t-1 \le (s-1)\lambda + t$, where $\lambda := \min\{E_n - E_m, y - H_m\} \ge 1$.

- $y = H_n - 1$. Note that $H_n - 1 = (n-1)(s+t)+(s-\delta_s)+n+1$ is even on account of $n \in \mathbb{Z}^{odd}$ and $s - \delta_s \in \mathbb{Z}^{even}$. Thus we move simply $y \to H'_n = 0$.

Case (iv) $x = n$ is even.

In this case, $H'_n = H_n - 1$ and so we have either $y > H_n$ or $0 \le y \le H_n - 2$.

For $y > H_n$. It is worth to note that if $s + t$ is odd, then $t + \delta_s$ is also odd, thereby $H_n = (s+t+1)n + 2 - t - \delta_s$ is odd; if $s+t$ is even, meaning that $t+\delta_s$ is also even, and so H_n is even. Therefore, if $(y, s+t \in \mathbb{Z}^{odd})$ or $(y, s+t \in \mathbb{Z}^{even})$, then we move $y \to H_n$ since $y - H_n$ is even; if $(y \in \mathbb{Z}^{odd}$ and $s+t \in \mathbb{Z}^{even})$ or $(y \in \mathbb{Z}^{even}$ and $s+t \in \mathbb{Z}^{odd})$, then we move $y \to H'_n$ because $y - H'_n = y - H_n + 1$ is still even.

For $0 \le y \le H_n - 2$. If $y = 0$, we move $(x, 0) \to (x-1, 0)$. If $1 \le y \le x - 1$, we move $(x, y) \to (x - y - 1 + \delta_y, 0) \in \mathscr{P}$ with $x - y - 1 + \delta_y = m$ being odd, which satisfies Eq. (1) with $k = y$ and $\ell = y + 1 - \delta_y$. Otherwise, analysis for $x \le y \le H_n - 2$ is the same as the proof of case (iii), more details are left to the reader. ■

Remark 1. Similar to Γ_{OE}, maybe we can define Γ_{EO}, *Even-Odd-Nim* (s,t)*-Wythoff's Game*: A player chooses the first pile and takes *even* $k > 0$ tokens, or chooses the second pile and takes *odd* $\ell > 0$ tokens, the move rules are the same with (s,t)-Wythoff's Game when move from both piles. The move rules of these two games imply that (x, y) is a P-position of Γ_{EO} if and only if (y, x) is a P-position of Γ_{OE}. Thus the P-positions of Γ_{EO} are easily obtained by Theorems 13, 14, 15 and 16 with (x, y) replaced by (y, x).

5 Conclusion

In this paper, Γ_K is completed solved for any $K, s, t \in \mathbb{Z}^+$ in both normal and misère play. It is a generalization of both the original (s,t)-Wythoff's game and EEW investigated in [12]. It is worth mentioning that both exponential and polynomial winning strategies for Γ_K are given in both normal and misère play. However, in misère play, whether Γ_K has a polynomial time winning strategy or not is still open for all $s > 1$.

Following this, Γ_{OA}, Γ_{OO}, and Γ_{OE} are investigated. Under both normal and misère play conventions, the sets of P-positions of these three games are given algebraically for all $s, t \ge 1$. Motivated by these games, we may associate more interesting games, for instance:

Open problem. Define Γ_{EE} (Even-Even-Nim (s,t)-Wythoff's game): a player may only remove an *even* (> 0) number of tokens when move from a single pile, and move rules remain unchanged when move from both piles. This game is also a rook type restriction of (s,t)-Wythoff's game. Then how to determine the P-positions of Γ_{EE}?

Further, what is the result if a player could remove a multiple of K ($\in \mathbb{Z}^+$) tokens when move from one pile (a generalization of Γ_{EE})? And what if a player is restricted to take $k \in \{nK + 1 : n \in \mathbb{Z}^+\}$ (or $k \in \{nK + K - 1 : n \in \mathbb{Z}^+\}$) tokens when move from one pile (it is a generalization of Γ_{OO}, which is precisely the case $K = 2$)?

Acknowledgement The authors are grateful to the anonymous referees for their valuable comments and suggestions, which have greatly improved the earlier version of this paper.

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