THE GAME OF END-WYTHOFF

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Abstract

Given a vector of finitely many piles of finitely many tokens. In End-Wythoff, two players alternate in taking a positive number of tokens from either end-pile, or taking the *same* positive number of tokens from both ends. The player first unable to move loses and the opponent wins. We characterize the P-positions (a_i, K, b_i) of the game for any vector K of middle piles, where a_i, b_i denote the sizes of the end-piles. A more succinct characterization can be made in the special case where K is a vector such that, for some $n \in \mathbb{Z}_{\geq 0}$, (K, n) and (n, K) are both P-positions. For this case the (noisy) initial behavior of the P-positions is described precisely. Beyond the initial behavior, we have $b_i - a_i = i$, as in the normal 2-pile Wythoff game.

Key Words: Combinatorial games, Wythoff's game, End-Wythoff's game, P-positions

1 Introduction

A position in the (impartial) game *End-Nim* is a vector of finitely many piles of finitely many tokens. Two players alternate in taking a positive number of tokens from either end-pile ("burning-the-candle-at-both-ends"). The player first unable to move loses and the opponent wins. Albert and Nowakowski [1] gave a winning strategy for End-Nim, by describing the P-positions of the game. (Their paper also includes a winning strategy for the partizan version of End-Nim.)

Wythoff's game [8] is played on two piles of finitely many tokens. Two players alternate in taking a positive number of tokens from a *single* pile, or taking the *same* positive number of tokens from both piles. The player first unable to move loses and the opponent wins. From among the many papers on this game, we mention just three: [2], [7], [3]. The P-positions (a'_i, b'_i) with $a'_i \leq b'_i$ of Wythoff's game have the property: $b'_i - a'_i = i$ for all $i \geq 0$.

Richard Nowakowski suggested to one of us (F) the game of *End-Wythoff*, whose positions are the same as those of End-Nim but with Wythoff-like moves allowed. Two players alternate in taking a positive number of tokens from either end-pile, or taking the *same* positive number of tokens from both ends. The player first unable to move loses and the opponent wins.

In this paper we charaterize the P-positions of End-Wythoff. Specifically, in Theorem 1 the P-positions (a_i, K, b_i) are given recursively for any vector of piles K.

The rest of the paper deals with values of K, deemed *special*, such that (n, K) and (K, n) are both P-positions for some $n \in \mathbb{Z}_{\geq 0}$. Theorem 3 gives a slightly cleaner recursive characterization than in the general case. In Theorems 4 and 5, the (noisy) initial behavior of the P-positions is described, and Theorem 6 shows that after the initial noisy behavior, we have $b_i - a_i = i$ as in the normal Wythoff game. Before all of that we show in Theorem 2 that if K is a P-position of End-Wythoff, then (a, K, b) is a P-position if and only if (a, b) is a P-position of Wythoff.

Finally, a polynomial algorithm is given for finding the P-positions (a_i, K, b_i) for any given vector of piles K.

2 P-Positions for General End-Wythoff Games

Definition 1. A *position* in the game of End-Wythoff is the empty game, which we denote by (0), or an element of $\bigcup_{i=1}^{\infty} \mathbb{Z}^i_{\geq 1}$, where we consider mirror images identical; that is, $(n_1, n_2, \ldots, n_k)$ and $(n_k, n_{k-1}, \ldots, n_1)$ are the same position.

Notation 1. For convenience of notation, we allow ourselves to insert extraneous 0s when writing a position. For example, $(0, K)$, $(K, 0)$, and $(0, K, 0)$ are all equivalent to K, and $(a, 0, b)$ is equivalent to (a, b).

Lemma 1. *Given any position K, there exist unique $l_K, r_K \in \mathbb{Z}_{\geq 0}$ such that (l_K, K) and (K, r_K) are P-positions.*

Proof. We phrase the proof for l_K, but the arguments hold symmetrically for r_K.

Uniqueness is fairly obvious: if (n, K) is a P-position and $m \neq n$, then (m, K) is not a P-position because we can move from one to the other.

For existence, if $K = (0)$, then $l_K = r_K = 0$, since the empty game is a P-position. Otherwise, let t be the size of the rightmost pile of K. If any of $(0, K), (1, K), \ldots, (2t, K)$ are P-positions, we are done. Otherwise, they are all N-positions. In this latter case, the moves that take $(1, K), (2, K), \ldots, (2t, K)$ to P-positions must all involve the rightmost pile. (That is, none of these moves take tokens only from the leftmost pile. Note that we cannot make this guarantee for $(0, K)$ because, for example, $(0, 2, 2) = (2, 2)$ can reach the P-position $(1, 2)$ by taking only from the leftmost pile.)

In general, if L is a position and $m < n$, then it cannot be that the same move takes both (m, L) and (n, L) to P-positions: if a move takes (m, L) to a P-position (m', L'), then that move takes (n, L) to $(m' + n - m, L')$, which is an N-position because we can move to (m', L').

In our case, however, there are only $2t$ possible moves that involve the rightmost pile: for $1 \leq i \leq t$, take i from the rightmost pile, or take i from both end-piles. We conclude that each of these moves takes one of $(1, K), (2, K), \ldots, (2t, K)$

to a P-position, so no move involving the rightmost pile can take $(2t+1, K)$ to a P-position. But also, no move that takes only from the leftmost pile takes $(2t+1, K)$ to a P-position because (n, K) is an N-position for $n < 2t+1$. Thus $(2t+1, K)$ cannot reach any P-position in one move, so it is a P-position, and $l_K = 2t+1$. □

We now state some definitions which will enable us to characterize P-positions as pairs at the 2 ends of a given vector K. For any subset $S \subset \mathbb{Z}_{\geq 0}$, $S \neq \mathbb{Z}_{\geq 0}$, let $\operatorname{mex} S = \min(\mathbb{Z}_{\geq 0} \setminus S) =$ least nonnegative integer not in S.

Definition 2. Let K be a position of End-Wythoff, and let $l = l_K$ and $r = r_K$ be as in Lemma 1. For $n \in \mathbb{Z}_{\geq 1}$, define

$$\begin{aligned} d_n &= b_n - a_n \\ A_n &= \{0, l\} \cup \{a_i : 1 \leq i \leq n-1\} \\ B_n &= \{0, r\} \cup \{b_i : 1 \leq i \leq n-1\} \\ D_n &= \{-l, r\} \cup \{d_i : 1 \leq i \leq n-1\}, \end{aligned}$$

where

$$a_n = \operatorname{mex} A_n \tag{1}$$

and b_n is the smallest number $x \in \mathbb{Z}_{\geq 1}$ satisfying both

$$x \notin B_n, \tag{2}$$

$$x - a_n \notin D_n. \tag{3}$$

Finally, let

$$A = \bigcup_{i=1}^{\infty} a_i \quad \text{and} \quad B = \bigcup_{i=1}^{\infty} b_i.$$

Note that the definitions of A and B ultimately depend only on the values of l and r. Thus, if K and L are positions with $l_K = l_L$ and $r_K = r_L$, then the pairs (a_i, b_i) that form P-positions when placed as end-piles around them will be the same.

Theorem 1.

$$P_K = \bigcup_{i=1}^{\infty} (a_i, K, b_i)$$

is the set of P-positions of the form (a, K, b) with $a, b \in \mathbb{Z}_{\geq 1}$.

Table 1: The first 15 outer piles of P-positions for some values of K.

	$K=(1,2)$			$K=(1,3)$			$K=(2,3)$			$K=(1,2,2)$		
	$l=0$		0	$l=4$		-4	$l=5$		-5	$l=1$		-1
		$r=0$	0		$r=1$	1		$r=3$	3		$r=1$	1
i	a_i	b_i	d_i	a_i	b_i	d_i	a_i	b_i	d_i	a_i	b_i	d_i
1	1	2	1	1	3	2	1	1	0	2	2	0
2	2	1	-1	2	2	0	2	4	2	3	5	2
3	3	5	2	3	6	3	3	2	-1	4	7	3
4	4	7	3	5	4	-1	4	5	1	5	3	-2
5	5	3	-2	6	10	4	6	10	4	6	10	4
6	6	10	4	7	5	-2	7	12	5	7	4	-3
7	7	4	-3	8	13	5	8	6	-2	8	13	5
8	8	13	5	9	15	6	9	15	6	9	15	6
9	9	15	6	10	7	-3	10	7	-3	10	6	-4
10	10	6	-4	11	18	7	11	18	7	11	18	7
11	11	18	7	12	20	8	12	8	-4	12	20	8
12	12	20	8	13	8	-5	13	21	8	13	8	-5
13	13	8	-5	14	23	9	14	23	9	14	23	9
14	14	23	9	15	9	-6	15	9	-6	15	9	-6
15	15	9	-6	16	26	10	16	26	10	16	26	10

Proof. Since moves are not allowed to alter the central piles of a position, any move from (a, K, b) with $a, b > 0$ will result in (c, K, d) with $c, d \geq 0$. Since $(l, K) = (l, K, 0)$ and $(K, r) = (0, K, r)$ are P-positions, they are the only P-positions with $c = 0$ or $d = 0$. Thus, to prove that P_K is the set of P-positions of the desired form, we must show that, from a position in P_K, one cannot reach (l, K), (K, r), or any position in P_K in a single move, and we must also show that from any $(a, K, b) \notin P_K$ with $a, b > 0$ there is a single move to at least one of these positions.

We begin by noting several facts about the sequences A and B.

(a) We see from (1) that $a_{n+1} = \begin{cases} a_n + 1, & \text{if } a_n + 1 \neq l \\ a_n + 2, & \text{if } a_n + 1 = l \end{cases}$ for $n \geq 1$, so A is strictly increasing.

(b) We can also conclude from (1) that $A = \mathbb{Z}_{\geq 1} \setminus \{l\}$.

(c) It follows from (2) that all elements in B are distinct. The same conclusion holds for A from (1).

We show first that from $(a_m, K, b_m) \in P_K$ one cannot reach any element of P_K in one move:

(i) $(a_m - t, K, b_m) = (a_n, K, b_n) \in P_K$ for some $0 < t \leq a_m$. Then $m \neq n$ but $b_m = b_n$, contradicting (c).

(ii) $(a_m, K, b_m - t) = (a_n, K, b_n) \in P_K$ for some $0 < t \leq b_m$. This implies that $a_m = a_n$, again contradicting (c).

(iii) $(a_m - t, K, b_m - t) = (a_n, K, b_n) \in P_K$ for some $0 < t \leq a_m$. Then $b_n - a_n = b_m - a_m$, contradicting (3).

It is a simple exercise to check that $(a_m, K, b_m) \in P_K$ cannot reach (l, K) or (K, r).

Now we prove that from $(a, K, b) \notin P_K$ with $a, b > 0$, there is a single move to (l, K), (K, r), or some $(a_n, K, b_n) \in P_K$.

If $a = l$, we can take all of the right-hand pile and reach (l, K). Similarly, if $b = r$, we can move to (K, r) by taking the left-hand pile.

Now assume $a \neq l$ and $b \neq r$. We know from (b) that $a \in A$, so let $a = a_n$. If $b > b_n$, then we can move to (a_n, K, b_n). Otherwise, $b < b_n$, so b must violate either (2) or (3).

If $b \in B_n$, then $b = b_m$ with $m < n$ (because $b \neq r$ and $b > 0$). Since $a_m < a_n$ by (a), we can move to (a_m, K, b_m) by drawing from the left pile.

If, on the other hand, $b - a_n \in D_n$, then there are three possibilities: if $b - a_n = b_m - a_m$ for some $m < n$, then we can move to (a_m, K, b_m) by taking $b - b_m = a_n - a_m > 0$ from both end-piles; if $b - a_n = -l$, then drawing $b = a_n - l$ from both sides puts us in (l, K); and if $b - a_n = r$, then taking $a_n = b - r$ from both sides leaves us with (K, r). □

3 P-positions for Special Positions

Examining Table 1 reveals a peculiarity that occurs when $l = r$.

Definition 3. A position K is *special* if $l_K = r_K$.

In such cases, if (a_i, b_i) occurs in a column, then (b_i, a_i) also appears in that column. Examples of special K are P-positions, where $l = r = 0$, and palindromes, where (l, K) is the unique P-position of the form (a, K), but $(K, r) = (r, K)$ is also a P-position, so $l = r$. However, other values of K can also be special. We saw $(1, 2)$—a P-position—and $(1, 2, 2)$ in Table 1; other examples are $(4, 1, 13), (7, 5, 15)$, and $(3, 1, 4, 10)$, to name a few.

We begin with the special case $l_K = r_K = 0$.

Theorem 2. *Let K be a P-position of End-Wythoff. Then (a, K, b) is a P-position of End-Wythoff if and only if (a, b) is a P-position of Wythoff.*

Proof. Induction on $a + b$, where the base $a = b = 0$ is obvious. Suppose the assertion holds for $a + b < t$, where $t \in \mathbb{Z}_{>0}$. Let $a + b = t$. If (a, b) is an N-position of Wythoff, then there is a move $(a, b) \to (a', b')$ to a P-position of Wythoff, so by induction (a', K, b') is a P-position hence (a, K, b) is an N-position. If, on the other hand, (a, b) is a P-position of Wythoff, then every follower (a', b') of (a, b) is an N-position of Wythoff, hence every follower (a', K, b') of (a, K, b) is an N-position, so (a, K, b) is a P-position. □

The remainder of this paper deals with other cases of special K. We will see that this phenomenon allows us to ignore the distinction between the left and the right side of K, which will simplify our characterization of the P-positions. We start this discussion by redefining our main terms accordingly. (Some of these definitions are not changed, but repeated for ease of reference.)

Definition 4. Let $r = r_k$, as above. For $n \in \mathbb{Z}_{\geq 1}$, define

$$\begin{aligned} d_n &= b_n - a_n, \\ A_n &= \{0, r\} \cup \{a_i : 1 \leq i \leq n-1\}, \\ B_n &= \{0, r\} \cup \{b_i : 1 \leq i \leq n-1\}, \\ V_n &= A_n \cup B_n, \\ D_n &= \{r\} \cup \{d_i : 1 \leq i \leq n-1\}, \end{aligned}$$

where

$$a_n = \operatorname{mex} V_n \tag{4}$$

and b_n is the smallest number $x \in \mathbb{Z}_{\geq 1}$ satisfying both

$$x \notin V_n, \tag{5}$$

$$x - a_n \notin D_n. \tag{6}$$

As before, $A = \bigcup_{i=1}^{\infty} a_i$ and $B = \bigcup_{i=1}^{\infty} b_i$.

With these definitions, our facts about the sequences A and B are somewhat different:

(A) The sequence A is strictly increasing because $1 \leq m < n \Longrightarrow a_n = \operatorname{mex} V_n > a_m$, since $a_m \in V_n$.

(B) It follows from (5) that all elements in B are distinct.

(C) Condition (5) also implies that $b_n \geq a_n = \operatorname{mex} V_n$ for all $n \geq 1$.

(D) $A \cup B = \mathbb{Z}_{\geq 1} \setminus \{r\}$ due to (4).

(E) $A \cap B$ is either empty or equal to $\{a_1\} = \{b_1\}$. First, note that $a_n \neq b_m$ for $n \neq m$, because $m < n$ implies that a_n is the mex of a set containing b_m by (4), and if $n < m$, then the same conclusion holds by (5). If $r = 0$, then $b_i - a_i \neq 0$ for all i, so $A \cap B = \emptyset$. Otherwise $r > 0$, and for $n = 1$, the minimum value satisfying (5) is $\operatorname{mex}\{0, r\} = a_1$, and in this case a_1 also satisfies (6); that is, $0 = a_1 - a_1 \notin \{r\}$. Therefore, $b_1 = a_1$, and $b_i - a_i \neq 0$ for $i > 1$, by (6).

Theorem 3. *If K is special, then*

$$P_K = \bigcup_{i=1}^{\infty} (a_i, K, b_i) \cup (b_i, K, a_i)$$

is the set of P-positions of the form (a, K, b) with $a, b \in \mathbb{Z}_{\geq 1}$.

Table 2: The first 20 outer piles of P-positions for some values of K. Note that B, while usually strictly increasing, need not always be, as illustrated at $K = (8, 6, 23)$, $i = 9$.

	$K = (0)$			$K = (1, 1, 2)$			$K = (5)$			$K = (8, 6, 23)$		
	$r = 0$			$r = 2$			$r = 3$			$r = 14$		
i	a_i	b_i	d_i	a_i	b_i	d_i	a_i	b_i	d_i	a_i	b_i	d_i
1	1	2	1	1	1	0	1	1	0	1	1	0
2	3	5	2	3	4	1	2	4	2	2	3	1
3	4	7	3	5	8	3	5	6	1	4	6	2
4	6	10	4	6	10	4	7	11	4	5	8	3
5	8	13	5	7	12	5	8	13	5	7	11	4
6	9	15	6	9	15	6	9	15	6	9	15	6
7	11	18	7	11	18	7	10	17	7	10	17	7
8	12	20	8	13	21	8	12	20	8	12	20	8
9	14	23	9	14	23	9	14	23	9	13	18	5
10	16	26	10	16	26	10	16	26	10	16	25	9
11	17	28	11	17	28	11	18	29	11	19	29	10
12	19	31	12	19	31	12	19	31	12	21	32	11
13	21	34	13	20	33	13	21	34	13	22	34	12
14	22	36	14	22	36	14	22	36	14	23	36	13
15	24	39	15	24	39	15	24	39	15	24	39	15
16	25	41	16	25	41	16	25	41	16	26	42	16
17	27	44	17	27	44	17	27	44	17	27	44	17
18	29	47	18	29	47	18	28	46	18	28	46	18
19	30	49	19	30	49	19	30	49	19	30	49	19
20	32	52	20	32	52	20	32	52	20	31	51	20

Table 2 lists the first few such (a_i, b_i) pairs for several special values of K. Note that the case $K = (0)$ corresponds to Wythoff's game.

Proof. As in the proof for general K, we need to show two things: from a position in P_K one cannot reach (r, K), (K, r), or any position in P_K in a single move, and from any $(a, K, b) \notin P_K$ with $a, b > 0$ there is a single move to at least one of these positions.

It is a simple exercise to see that one can reach neither (r, K) nor (K, r) from $(a_m, K, b_m) \in P_K$, so we show that it is impossible to reach any position in P_K in one move:

(i) $(a_m - t, K, b_m) \in P_K$ for some $0 < t \leq a_m$. We cannot have $(a_m - t, K, b_m) = (a_n, K, b_n)$ because it contradicts (B). If $(a_m - t, K, b_m) = (b_n, K, a_n)$, then $a_n = b_m$, so $m = n = 1$ by (E). But then $a_m - t = b_n = a_m$, a contradiction.

(ii) $(a_m, K, b_m - t) \in P_K$ for some $0 < t \leq b_m$. This case is symmetric to (i).

(iii) $(a_m - t, K, b_m - t) \in P_K$ for some $0 < t \le a_m$. We cannot have $(a_m - t, K, b_m - t) = (a_n, K, b_n)$ because it contradicts (6). If $(a_m - t, K, b_m - t) = (b_n, K, a_n)$, then $b_m - a_m = -(b_n - a_n)$. But (C) tells us that $b_m - a_m \ge 0$ and $b_n - a_n \ge 0$, so $b_m - a_m = b_n - a_n = 0$, contradicting (6).

Similar reasoning holds if one were starting from $(b_m, K, a_m) \in P_K$.

Now we prove that from $(a, K, b) \notin P_K$ with $a, b > 0$ there is a single move to (r, K), to (K, r), to some $(a_n, K, b_n) \in P_K$, or to some $(b_n, K, a_n) \in P_K$. We assume that $a \le b$, but the arguments hold symmetrically for $b \le a$.

If $a = r$, we can move to (r, K) by taking the entire right-hand pile. Otherwise, by (D), a is in either A or B. If $a = b_n$ for some n, then $b \ge a = b_n \ge a_n$. Since $(a, K, b) \notin P_K$, we have $b > a_n$, so we can move b to a_n, thereby reaching $(b_n, K, a_n) \in P_K$.

If $a = a_n$ for some n, then if $b > b_n$, we can move to $(a_n, K, b_n) \in P_K$. Otherwise we have, $a = a_n \le b < b_n$. We consider 2 cases.

I. $b - a_n \in D_n$. If $b - a_n = r$, then we can take $b - r = a_n$ from both ends to reach (K, r). Otherwise, $b - a_n = b_m - a_m$ for some $m < n$, and $b - b_m = a_n - a_m > 0$ since $a_n > a_m$ by (A). Thus we can move to $(a_m, K, b_m) \in P_K$ by taking $a_n - a_m = b - b_m$ from both a_n and b.

II. $b - a_n \notin D_n$. This shows that b satisfies (6). Since $b < b_n$ and b_n is the smallest value satisfying both (5) and (6), we must have $b \in V_n$. By assumption, $b > 0$. If $b = r$, then we can move to (K, r) by taking the entire left-hand pile. Otherwise, since $b \ge a_n > a_m$ for all $m < n$, it must be that $b = b_m$ with $m < n$. We now see from (A) that $a_m < a_n$, so we can draw from the left-hand pile to obtain $(a_m, K, b_m) \in P_K$. □

Lemma 2. *For $m, n \in \mathbb{Z}_{\ge 1}$, if $\{0, \dots, m-1\} \subseteq D_n$, $m \notin D_n$ and $a_n + m \notin V_n$, then $b_n = a_n + m$ and $\{0, \dots, m\} \subseteq D_{n+1}$.*

Proof. We have $x < a_n \implies x \in V_n$, and $a_n \le x < a_n + m \implies x - a_n \in D_n$, so no number smaller than $a_n + m$ satisfies both (5) and (6). The number $a_n + m$, however, satisfies both since, by hypothesis, $a_n + m \notin V_n$ and $m \notin D_n$, so $b_n = a_n + m$. Since $b_n - a_n = m$, $\{0, \dots, m\} \subseteq D_{n+1}$. □

Lemma 3. *For $m \in \mathbb{Z}_{\ge 1}$, if $D_m = \{0, \dots, m-1\}$, then $b_n = a_n + n$ for all $n \ge m$.*

Proof. We see that $m \notin D_m$. Also, $a_m + m \notin V_m$: it cannot be in A_m because A is strictly increasing, and it cannot be in B_m because if it were, we would get $m = b_i - a_m < b_i - a_i \in D_m$, a contradiction. So Lemma 2 applies, and $b_m = a_m + m$.

This shows that $D_{m+1} = \{0, \dots, m\}$, so the result follows by induction. □

Lemma 4. *If $1 \le m \le r < a_m + m - 1$ and $D_m = \{r, 0, 1, \dots, m-2\}$, then $d_m = m - 1$. Thus, for $m \le n \le r$, $d_n = n - 1$.*

Proof. For $0 < i < m$ we have $a_i < a_m$ by (A) and $d_i < m-1$ since we cannot have $d_i = r$. Hence $a_m + m - 1 > a_i + d_i = b_i \geq a_i$. Also by hypothesis, $a_m + m - 1 > r$, so $a_m + m - 1 \notin V_m$. Since $m-1 \notin D_m$, Lemma 2 (with $n = m$ and $m = m-1$) implies $d_m = m-1$.

For $m \leq n \leq r$, the condition in the lemma holds inductively, so the conclusion holds, as well. □

We will now begin to note further connections between the P-positions in End-Wythoff and those in standard Wythoff's Game, to which end we introduce some useful notation.

Notation 2. The P-positions of Wythoff's game—i.e., the 2-pile P-positions of End-Wythoff, along with $(0,0) = (0)$—are denoted by $\bigcup_{i=0}^{\infty}(a'_i, b'_i)$, where $a'_n = \lfloor n\phi \rfloor$ and $b'_n = \lfloor n\phi^2 \rfloor$ for all $n \in \mathbb{Z}_{\geq 0}$, and $\phi = (1+\sqrt{5})/2$ is the golden ratio. We write $A' = \bigcup_{i=0}^{\infty} a'_i$ and $B' = \bigcup_{i=0}^{\infty} b'_i$.

An important equivalent definition of A' and B' is, for all $n \in \mathbb{Z}_{\geq 0}$ (see [3]),

$$\begin{aligned} a'_n &= \operatorname{mex}\{a'_i, b'_i : 0 \leq i \leq n-1\}, \\ b'_n &= a'_n + n. \end{aligned}$$

The following is our main lemma for the proof of Theorem 4.

Lemma 5. *Let $n \in \mathbb{Z}_{\geq 0}$. If $a'_n + 1 < r$, then $a_{n+1} = a'_n + 1$. If $b'_n + 1 < r$, then $b_{n+1} = b'_n + 1$.*

Proof. Note that $a'_0 + 1 = b'_0 + 1 = 1$. If $1 < r$, then $a_1 = \operatorname{mex}\{0, r\} = 1$, and 1 satisfies both (5) and (6), so $b_1 = 1$. So the result is true for $n = 0$.

Assume that the lemma's statement is true for $0 \leq i \leq n-1$ $(n \geq 1)$, and assume further that $a'_n + 1 < r$. Then $a_0 = 0 < a'_n + 1$. Also, $a'_i + 1 < a'_n + 1 < r$ for $0 \leq i \leq n-1$ because A' is strictly increasing. But, $a_{i+1} = a'_i + 1$ for $0 \leq i \leq n-1$ by the induction hypothesis, so $a_i < a'_n + 1$ for $1 \leq i \leq n$. Thus, we have shown that $a'_n + 1 \notin A_{n+1}$.

Let m be the least index such that $b'_m + 1 \geq r$, and let $j = \min\{m, n\}$. Then $b'_{i-1} + 1 < r$ for $1 \leq i \leq j$, so $b_i = b'_{i-1} + 1$ by the induction hypothesis. We know that $b'_r = a'_s \implies r = s = 0$, so $b'_{i-1} \neq a'_n$ because $n \geq 1$. Therefore $b_i = b'_{i-1} + 1 \neq a'_n + 1$, so $a'_n + 1 \notin B_{j+1}$.

If $j = n$, then we have shown that $a'_n + 1 \notin V_{n+1}$. Otherwise, $j = m$. For $i \geq m+1$ we have $d_i \geq m$ by (6), since $d_i = b_i - a_i = b'_{i-1} + 1 - (a'_{i-1} + 1) = i-1$ for $1 \leq i \leq m$ by our induction hypothesis. Also, $a_i \geq a_{m+1}$ for $i \geq m+1$, by (A). Therefore, for $i \geq m+1$, $b_i = a_i + d_i \geq a_{m+1} + m = (a'_m + 1) + m = b'_m + 1 \geq r > a'_n + 1$. Thus we see that $a'_n + 1 \notin \{b_i : i \geq m+1\}$, and we have shown that $a'_n + 1 \notin V_{n+1}$.

Now, $0 \in V_{n+1}$, and if $1 \leq x < a'_n + 1$, then $0 \leq x - 1 < a'_n$, so $x - 1 \in \{a'_i, b'_i : 0 \leq i < n\}$. Thus, for some i with $0 \leq i < n$, either $x = a'_i + 1 = a_{i+1}$ or $x = b'_i + 1 = b_{i+1}$ by the induction hypothesis, so $x \in V_{n+1}$. Hence $a'_n + 1 = \operatorname{mex} V_{n+1}$. This proves the first statement of the lemma: $a_{n+1} = \operatorname{mex} V_{n+1} = a'_n + 1$.

Note that if $b'_i + 1 < r$ for some $i \in \mathbb{Z}_{\geq 0}$, then a fortiori $a'_i + 1 < r$. Hence by the first part of the proof, $a_{i+1} = a'_i + 1$. Thus,

$$b'_i + 1 < r \implies a_{i+1} = a'_i + 1. \tag{7}$$

For the second statement of the lemma, assume that the result is true for $0 \leq i \leq n-1$ $(n \geq 1)$, and that $b'_n + 1 < r$. Then, for $0 \leq i \leq n-1$, we know $a_{i+1} = a'_i + 1$ by (7), and $b_{i+1} = b'_i + 1$ by the induction assumption. Therefore $d_i = i - 1$ for $1 \leq i \leq n$, so b_{n+1} cannot be smaller than $a_{n+1} + n$. Also $a_{n+1} = a'_n + 1$ by (7).

Consider $a_{n+1} + n = a'_n + 1 + n = b'_n + 1$. We have $0 < b'_n + 1 < r$, and for $1 \leq i \leq n$, $a_i \leq b_i = b'_{i-1} + 1 < b'_n + 1$. This implies that $b'_n + 1 \notin V_{n+1}$, and we conclude that $b_{n+1} = b'_n + 1$. □

Corollary 1. *Let $n \in \mathbb{Z}_{\geq 0}$. If $a_{n+1} < r$, then $a_{n+1} = a'_n + 1$. If $b_{n+1} < r$, then $b_{n+1} = b'_n + 1$.*

Proof. Note that $a_{n+1} > 0$. Since $A' \cup B' = \mathbb{Z}_{\geq 0}$, either $a_{n+1} = a'_i + 1$ or $a_{n+1} = b'_i + 1$. If $a'_i + 1 = a_{n+1} < r$, then $a_{n+1} = a'_i + 1 = a_{i+1}$ by Lemma 5, so $i = n$. If $b'_i + 1 = a_{n+1} < r$, then $a_{n+1} = b'_i + 1 = b_{i+1}$ by Lemma 5, so $i = n = 0$ by (E), and $a_1 = b_1 = b'_0 + 1 = a'_0 + 1$. The same argument holds for b_{n+1}. □

Corollary 2. *For $1 \leq n \leq r-1$, $n \in A$ if and only if $n - 1 \in A'$ and $n \in B$ if and only if $n - 1 \in B'$.*

Proof. This follows from Lemma 5 and Corollary 1. □

Theorem 4. *If $r = a'_n + 1$, then for $1 \leq i \leq r$, $d_i = i - 1$. Furthermore, for $1 \leq i \leq n$, $a_i = a'_{i-1} + 1$ and $b_i = b'_{i-1} + 1$.*

Proof. If $r = 1$, then $n = 0$. In this case, note that $a_1 = b_1 = 2$, so $d_1 = 0$, and that the second assertion of the theorem is vacuously true.

Otherwise, $r \geq 2$, and we again let m be the least index such that $b'_m + 1 \geq r$. Note that $m \geq 1$ because $b'_0 + 1 = 1 < r$. Thus $b'_m \neq a'_n = r - 1$, so in fact $b'_m + 1 > r$. For $1 \leq i \leq m$, since $a'_{i-1} \leq b'_{i-1}$ and B' is increasing, we have $a'_{i-1} + 1 \leq b'_{i-1} + 1 \leq b'_{m-1} + 1 < r$, so $a_i = a'_{i-1} + 1$ and $b_i = b'_{i-1} + 1$ by Lemma 5. We see that $d_i = i - 1$ for $1 \leq i \leq m$, so $D_{m+1} = \{r, 0, \ldots, m-1\}$.

Notice that $a_{m+1} + m > r$ because either $a_{m+1} > r$ and the fact is clear, or $a_{m+1} < r$, so $a_{m+1} = a'_m + 1$ by Corollary 1, which implies that $a_{m+1} + m = a'_m + 1 + m = b'_m + 1 > r$. Also, $m + 1 \leq b'_{m-1} + 2$ (because $1 + 1 = b'_0 + 2$ and B' is strictly increasing) and $b'_{m-1} + 1 < r$, so $m + 1 \leq b'_{m-1} + 2 \leq r$. We can now invoke Lemma 4 to see that $d_i = i - 1$ for $m + 1 \leq i \leq r$, so we have $d_i = i - 1$ for $1 \leq i \leq r$.

Since $n \leq a'_n < r$, in particular $d_i = i - 1$ for $1 \leq i \leq n$. With i in this range, we know $a'_{i-1} + 1 < a'_n + 1 = r$, so we get $a_i = a'_{i-1} + 1$ by Lemma 5, and since $d_i = i - 1$, $b_i = a_i + i - 1 = a'_{i-1} + 1 + i - 1 = b'_{i-1} + 1$. □

Theorem 5. *If $r = b'_n + 1$, then for $1 \leq i \leq r$, $d_i = i - 1$ except as follows:*

- *If $n = 0$, there are no exceptions.*
- *If $a'_n + 1 \in B'$, then $d_{n+1} = n + 1$ and $d_{n+2} = n$.*
- *If $n = 2$, then $d_3 = 3$, $d_4 = 4$ and $d_5 = 2$.*
- *Otherwise, $d_{n+1} = n + 1$, $d_{n+2} = n + 2$, $d_{n+3} = n + 3$, and $d_{n+4} = n$.*

Proof. One can easily verify the theorem for $0 \leq n \leq 2$—that is, when $r = 1$ (first bullet), 3 (second bullet), or 6 (third bullet). So we assume $n \geq 3$.

Lemma 5 tells us that for $1 \leq i \leq n$, $a_i = a'_{i-1} + 1$ and $b_i = b'_{i-1} + 1$ because $a'_{i-1} + 1 \leq b'_{i-1} + 1 < b'_n + 1 = r$. This implies that $d_i = i - 1$ for $1 \leq i \leq n$. This is not the case for d_{n+1}: $a'_n + 1 < b'_n + 1 = r$, so $a_{n+1} = a'_n + 1$, but $a_{n+1} + n = b'_n + 1 = r$, which cannot be b_{n+1}. We must have $b_{n+1} \geq a_{n+1} + n$, however, and $a_i \leq b_i < r = a_{n+1} + n$ for $1 \leq i \leq n$, so we see that $a_{n+1} + n + 1 \notin V_{n+1}$; thus $b_{n+1} = a_{n+1} + n + 1 = a'_n + n + 2 = b'_n + 2$, and $d_{n+1} = n + 1$.

If $a'_n + 1 \in B'$, then $a'_{n+1} = a'_n + 2$ (because B' does not contain consecutive numbers) and $a'_{n+1} + 1 = a'_n + 3 \leq a'_n + n = b'_n = r - 1$, so Lemma 5 tells us that $a_{n+2} = a'_{n+1} + 1 = a'_n + 3$. Now, $a_{n+2} + n = a'_n + n + 3 > a'_n + n + 2 = b_{n+1} \geq b_i$ for all $i \leq n + 1$, so $b_{n+2} = a_{n+2} + n$, and we see $d_{n+2} = n$. That is, we have $a_{n+1}, b_i, a_{n+2}, \ldots, r, b_{n+1}, b_{n+2}$.

This gives us $D_{n+3} = \{r, 0, \ldots, n + 1\}$. Also, $5 < b'_2 + 1 = 6$ so, since B' is strictly increasing and $n + 3 \geq 5$, we know $n + 3 < b'_n + 1 = r$. Furthermore, $r < b_{n+1} = a_{n+1} + n + 1 < a_{n+3} + n + 2$. Therefore, we can cite Lemma 4 to assert that $d_i = i - 1$ for $n + 3 \leq i \leq r$.

If, on the other hand, $a'_n + 1 \notin B'$, then $a'_n + 1 = a'_{n+1}$. Note that $a'_3 + 1 = 5 = b'_2$ and $a'_4 + 1 = 7 = b'_3$, so we can assume $n \geq 5$. We have $a'_{n+1} + 1 = a'_n + 2 < a'_n + n = b'_n < r$, so $a_{n+2} = a'_{n+1} + 1 = a'_n + 2$, and we find that $a_{n+2} + n = a'_n + n + 2 = b_{n+1} \in V_{n+2}$. Also, a difference of $n + 1$ already exists, but $a_{n+2} + n + 2$ is not in V_{n+2}, as it is greater than all of the previous B values. So we get $b_{n+2} = a_{n+2} + n + 2$, and $d_{n+2} = n + 2$. We have the following picture: $a_{n+1}, a_{n+2}, \ldots, r, b_{n+1}, -, b_{n+2}$.

Now, since $a'_n + 1 \in A'$, $a'_n + 2$ must be in B' because A' does not contain three consecutive values. Because $a'_n + 3 \leq a'_n + n = b'_n = r - 1$, we have $a_{n+2} + 1 = a'_n + 3 \in B$ by Corollary 2. Also, $a'_n + 3 \in A'$ because B' does not contain consecutive values, and $a'_n + 4 \leq r - 1$, so $a'_n + 4 \in A$. We therefore have $a_{n+1}, a_{n+2}, b_j, a_{n+3}, \ldots, r, b_{n+1}, -, b_{n+2}$. Since $a_{n+3} + n = b_{n+2}$ and differences of $n+1$ and $n+2$ already occurred, we get $b_{n+3} = a_{n+3} + n + 3$, and $d_{n+3} = n + 3$, and the configuration is $a_{n+1}, a_{n+2}, b_j, a_{n+3}, \ldots, r, b_{n+1}, -, b_{n+2}, -, -, b_{n+3}$.

If $n = 5$, then $r = b'_5 + 1 = 14$, and one can check that $a_{n+3} = a_8 = 12$ and $a_{n+4} = a_9 = 13 = a_{n+3} + 1$. If $n \geq 6$, then $a_{n+3} + 2 = a_{n+1} + 5 \leq a_{n+1} + n - 1 = r - 1$. The sequence B' does not contain consecutive values, so either $a_{n+3} \in A'$ or $a_{n+3} + 1 \in A'$, and therefore either $a_{n+3} + 1 \in A$ or $a_{n+3} + 2 \in A$. So regardless of the circumstances, either $a_{n+4} = a_{n+3} + 1$ or $a_{n+4} = a_{n+3} + 2$.

This means that either $a_{n+4} + n = a_{n+3} + n + 1 = b_{n+2} + 1$ or $a_{n+4} + n = a_{n+3} + n + 2 = b_{n+2} + 2$. In either case, this spot is not taken by an earlier b_i, so $b_{n+4} = a_{n+4} + n$, and $d_{n+4} = n$.

A few moments of reflection reveal that $4 \leq a_3$. Since A is strictly increasing, this gives us that $5 \leq a_4$ and, in general, $n+5 \leq a_{n+4}$. We now have $n+5 \leq a_{n+4} < r < b_{n+1} = a_{n+1}+n+1 < a_{n+5}+n+4$, and $D_{n+5} = \{r, 0, \dots, n+3\}$, so Lemma 4 completes the proof. □

Theorem 6. *If $n \geq r+1$, then $d_n = n$.*

Proof. The smallest n which fall under each of the bullets of Theorem 5 are $n=0$, $n=1$, $n=2$, and $n=5$, respectively. ($n=3$ and $n=4$ fall under the second bullet.) Notice that $n+2 \leq b'_n+1$ when $n \geq 1$ since $1+2 \leq b'_1+1 = 3$ and B' is strictly increasing. Similarly, $n+3 \leq b'_n+1$ when $n \geq 2$ since $2+3 \leq b'_2+1 = 6$, and $n+4 \leq b'_n+1$ when $n \geq 3$ because $3+4 \leq b'_3+1 = 8$. Therefore, we see that all of the exceptions mentioned in Theorem 5 occur before index $r+1 = b'_n+2$.

Theorems 4 and 5, combined with this observation, reveal that $D_{r+1} = \{0, \dots, r\}$, whether $r = a'_n+1$ or $r = b'_n+1$. Thus, by Lemma 3, $d_n = n$ for $n \geq r+1$. □

4 Generating P-positions in Polynomial Time

Any position of End-Wythoff is specified by a vector whose components are the pile sizes. We consider K to be a constant. The input size of a position (a, K, b) is thus $O(\log a + \log b)$. We seek an algorithm polynomial in this size.

Theorem 6 shows that we can express A and B beyond r as

$$\begin{aligned} a_n &= \operatorname{mex}(X \cup \{a_i, b_i : r+1 \leq i < n\}), \quad && n \geq r+1, \\ b_n &= a_n + n, && n \geq r+1, \end{aligned}$$

where $X = V_{r+1}$. This characterization demonstrates that the sequences generated from special End-Wythoff positions are a special case of those studied in [4], [5], [6]. In [4] it is proved that $a'_n - a_n$ is eventually constant except for certain "subsequences of irregular shifts", each of which obeys a Fibonacci recurrence. That is, if i and j are consecutive indices within one of these subsequences of irregular shifts, then the next index in the subsequence is $i+j$. This is demonstrated in Figure 1.

Relating our sequences to those of [4] is useful because that paper's proofs give rise to a polynomial algorithm for computing the values of the A and B sequences in the general case dealt with there. For the sake of self-containment, we begin by introducing some of the notation used there and mention some of the important theorems and lemmas.

Definition 5. Let $c \in \mathbb{Z}_{\geq 1}$. (For Wythoff's game, $c = 1$.)

$$\begin{aligned} a'_n &= \operatorname{mex}\{a'_i, b'_i : 1 \leq i < n\}, \quad n \geq 1; \\ b'_n &= a'_n + cn, \quad n \geq 1; \\ m_0 &= \min\{m : a_m > \max(X)\}; \end{aligned}$$

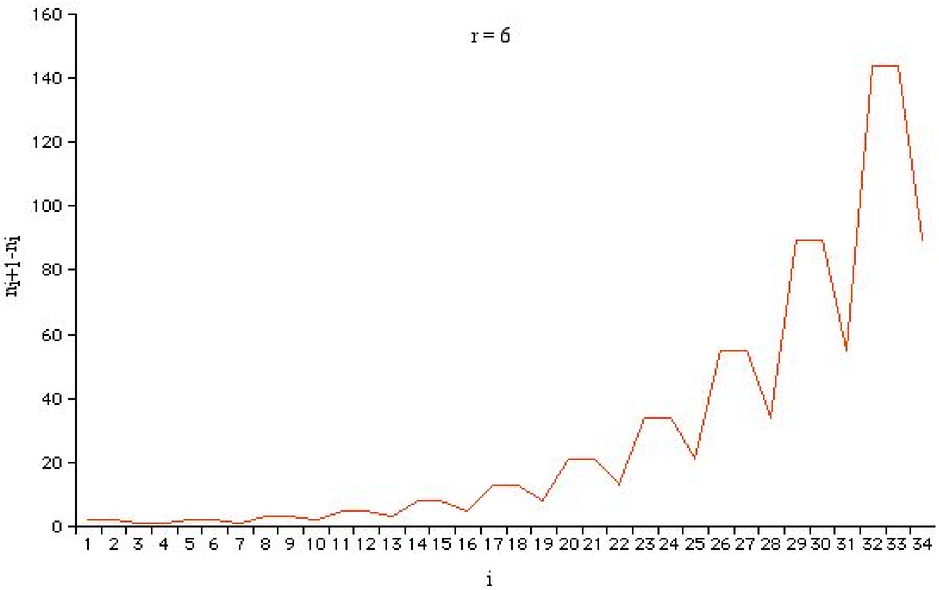

Figure 1: With $r = 6$, the distance between consecutive indices of P-positions which differ from Wythoff's game's P-positions. (That is, n_i is the subsequence of indices where $(a_n, b_n) \neq (a'_n, b'_n)$.) Note that every third point can be connected to form a Fibonacci sequence.

$$\begin{aligned} s_n &= a'_n - a_n, \quad n \geq m_0; \\ \alpha_n &= a_{n+1} - a_n, \quad n \geq m_0; \\ \alpha'_n &= a'_{n+1} - a'_n, \quad n \geq 1; \\ W &= \{\alpha_n\}_{n=m_0}^{\infty}; \\ W' &= \{\alpha'_n\}_{n=1}^{\infty}. \end{aligned}$$

$F : \{1,2\}^* \to \{1,2\}^*$ is the non-erasing morphism

$$F : \begin{array}{ccc} 2 & \to & 1^c 2 \\ 1 & \to & 1^{c-1} 2 \end{array} .$$

A *generator* for W (W') is a word of the form $u = \alpha_t \cdots \alpha_{n-1}$ ($u' = \alpha'_r \cdots \alpha'_{m-1}$), where $a_n = b_t + 1$ ($a'_m = b'_r + 1$). We say that W, W' are generated *synchronously* if there exist generators u, u', such that $u = \alpha_t \cdots \alpha_{n-1}, u' = \alpha'_t \cdots \alpha'_{n-1}$ (same indices t, n), and

$$\forall k \geq 0, F^k(u) = \alpha_g \cdots \alpha_{h-1} \Longleftrightarrow F^k(u') = \alpha'_g \cdots \alpha'_{h-1},$$

where $a_h = b_g + 1$.

A *well-formed string of parentheses* is a string $\vartheta = t_1 \cdots t_n$ over some alphabet which includes the letters '(', ')', such that for every prefix μ of ϑ, $|\mu|_(\geq |\mu|_)$

(never close more parentheses than were opened), and $|\vartheta|_(= |\vartheta|_)$ (don't leave opened parentheses).

The *nesting level* $N(\vartheta)$ of such a string is the maximal number of opened parentheses: let $p_1, \ldots, p_n$ satisfy

$$p_i = \begin{cases} 1 & \text{if } t_i = (\\ -1 & \text{if } t_i =) \\ 0 & \text{otherwise} \end{cases},$$

then

$$N(\vartheta) = \max_{1 \le k \le n} \left\{ \sum_{i=1}^{k} p_i \right\}.$$

With these definitions in mind, we cite the theorems, lemmas, and corollaries necessary to explain our polynomial algorithm.

Theorem 7. *There exist $p \in \mathbb{Z}_{\ge 1}, \gamma \in \mathbb{Z}$, such that, either for all $n \ge p$, $s_n = \gamma$; or else, for all $n \ge p$, $s_n \in \{\gamma - 1, \gamma, \gamma + 1\}$. If the second case holds, then:*

1. *s_n assumes each of the three values infinitely often.*
2. *If $s_n \ne \gamma$ then $s_{n-1} = s_{n+1} = \gamma$.*
3. *There exists $M \in \mathbb{Z}_{\ge 1}$, such that the indices $n \ge p$ with $s_n \ne \gamma$ can be partitioned into M disjoint sequences, $\{n_j^{(i)}\}_{j=1}^{\infty}, i = 1, \ldots, M$. For each of these sequences, the shift value alternates between $\gamma - 1$ and $\gamma + 1$:*

$$\begin{aligned} s_{n_j^{(i)}} = \gamma + 1 &\implies s_{n_{j+1}^{(i)}} = \gamma - 1; \\ s_{n_j^{(i)}} = \gamma - 1 &\implies s_{n_{j+1}^{(i)}} = \gamma + 1. \end{aligned}$$

Theorem 8. *Let $\{n_j\}_{j=1}^{\infty}$ be one of these subsequences of irregular shifts. Then it satisfies the following recurrence:*

$$\forall j \ge 3, n_j = cn_{j-1} + n_{j-2}.$$

Corollary 3. *If for some $t \ge m_0, b_t + 1 = a_n$ and $b'_t + 1 = a'_n$, then the words*

$$\begin{aligned} u &= \alpha_t \cdots \alpha_{n-1}, \\ u' &= \alpha'_t \cdots \alpha'_{n-1}, \end{aligned}$$

are permutations of each other.

Lemma 6 (Synchronization Lemma)**.** *Let m_1 be such that $a_{m_1} = b_{m_0} + 1$. Then there exists an integer $t \in [m_0, m_1]$, such that $b_t + 1 = a_n$ and $b'_t + 1 = a'_n$.*

Corollary 4. *If for some $t \ge m_0$, $b_t + 1 = a_n$ and $b'_t + 1 = a'_n$, then W, W' are generated synchronously by u, u', respectively.*

In comparing u and u', it will be useful to write them in the following form:

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} \alpha_t \cdots \alpha_{n-1} \\ \alpha'_t \cdots \alpha'_{n-1} \end{bmatrix},$$

and we will apply F to these pairs: $F\left(\begin{bmatrix} u \\ u' \end{bmatrix}\right) := \begin{bmatrix} F(u) \\ F(u') \end{bmatrix}$. Since u, u' are permutations of each other by Corollary 3, if we write them out in this form, then the columns $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ occur the same number of times. Thus we can regard $\begin{bmatrix} u \\ u' \end{bmatrix}$ as a well-formed string of parentheses: put '$\bullet$' for $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and put '(','')' for $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ alternately such that the string remains well-formed. That is, if the first non-equal pair we encounter is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then '(' stands for $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and ')' stands for $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ until all opened parentheses are closed. Then we start again, by placing '(' for the first occurrence different from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Example 1.

$$\begin{bmatrix} 122 \\ 221 \end{bmatrix} \longrightarrow (\bullet), \quad \begin{bmatrix} 1221 \\ 2112 \end{bmatrix} \longrightarrow ()(), \quad \begin{bmatrix} 22211211 \\ 21112122 \end{bmatrix} \longrightarrow \bullet((\bullet)()).$$

Lemma 7 (Nesting Lemma)**.** *Let $u(0) \in \{1,2\}^*$, and let $u'(0)$ be a permutation of $u(0)$. If $c = 1$ and $u(0)$ or $u'(0)$ contains 11, put $u := F(u(0)), u' := F(u'(0))$. Otherwise, put $u := u(0), u' := u'(0)$. Let $\vartheta \in \{\bullet,\ (,\ )\}^*$ be the parentheses string of $\begin{bmatrix} u \\ u' \end{bmatrix}$. Then successive applications of F decrease the nesting level to* 1*. Specifially,*

- **(I)** *If $c > 1$, then $N(\vartheta) > 1 \Longrightarrow N(F(\vartheta)) < N(\vartheta)$.*
- **(II)** *If $c = 1$,*
  - **(a)** $N(\vartheta) > 2 \Longrightarrow N(F(\vartheta)) < N(\vartheta)$*;*
  - **(b)** $N(\vartheta) = 2 \Longrightarrow N(F^2(\vartheta)) = 1$*.*

Lemma 8. *Under the hypotheses of the previous lemma, if $N(\vartheta) = 1$, then $F^2(\vartheta)$ has the form*

$$\cdots()\cdots()\cdots()\cdots, \tag{8}$$

where the dot strings consist of '$\bullet$' letters and might be empty. Further applications of F preserve this form, with the same number of parentheses pairs; the only change is that the dot strings grow longer.

We now have the machinery necessary to sketch the polynomial algorithm for generating the sequences A and B. There is a significant amount of initial computation, but then we can use the Fibonacci recurrences from Theorem 8 to obtain any later values for s_n, and thus for a_n and b_n as well. Here are the initial computations:

- Compute the values of A and B until $a_n = b_t + 1$ and $a'_n = b'_t + 1$. The Synchronization Lemma assures us that we can find such values with $m_0 \le t \le m_1$, where m_1 is the index such that $a_{m_1} = b_{m_0} + 1$. Corollary 4 tells us that W, W' are generated synchronously by $u = \alpha_t \dots \alpha_{n-1}, u' = \alpha'_t \dots \alpha'_{n-1}$.

- Iteratively apply F to u and u' until the parentheses string of $w = F^k(u)$ and $w' = F^k(u')$ is of the form (8). We know this will eventually happen because of Lemmas 7 and 8.

- Let p and q be the indices such that $w = \alpha_p \dots \alpha_q$ and $w' = \alpha'_p \dots \alpha'_q$. Compute A up to index p, and let $\gamma = a'_p - a_p$. At this point, noting the differences between w and w' gives us the initial indices for the subsequences of irregular shifts. Specifically, if letters $i, i+1$ of the parentheses string of $\begin{bmatrix} w \\ w' \end{bmatrix}$ are '()', then $i+1$ is an index of irregular shift. Label these indices of irregular shifts $n_1^{(1)}, \dots, n_1^{(M)}$ and, for $1 \le i \le M$, let $o_i = s_{n_1^{(i)}} - \gamma \in \{-1, 1\}$. (The o_i indicate whether the ith subsequence of irregular shifts begins offset by $+1$ or by -1 from the regular shift, γ.)

- Apply F once more to w and w'. The resulting sequences will again have M pairs of indices at which $w' \ne w$; label the indices of irregular shifts $n_2^{(1)}, \dots, n_2^{(M)}$.

With this initial computation done, we can determine a_n and b_n for $n \ge n_1^{(1)}$ as follows: for each of the M subsequences of irregular shifts, compute successive terms of the subsequence according to Theorem 8 until reaching or exceeding n. That is, for $1 \le i \le M$, compute $n_1^{(i)}, n_2^{(i)}, \dots$ until $n_j^{(i)} \ge n$. Since the $n_j^{(i)}$ are Fibonacci-like sequences, they grow exponentially, so they will reach or exceed the value n in time polynomial in $\log n$. If we obtain $n = n_j^{(i)}$ for some i, j, then

$$s_n = \begin{cases} \gamma + o_i, & \text{if } j \text{ is odd} \\ \gamma - o_i, & \text{if } j \text{ is even} \end{cases},$$

since each subsequence alternates being offset by $+1$ and by -1, by Theorem 7. If, on the other hand, every subsequence of irregular shifts passes n without having a term equal n, then $s_n = \gamma$. Once we know s_n, we have $a_n = a'_n - s_n$ and $b_n = a_n + n$. This implies $b_{n+1} - b_n \in \{2, 3\}$, hence the mex function implies $a_{n+1} - a_n \in \{1, 2\}$. Therefore $a_n \le 2n$, and the algorithm is polynomial.

In the case of sequences deriving from special positions of End-Wythoff, we must compute the value of r before we can begin computing A and B. After that, the initial computation can be slightly shorter than in the general case, as we are about to see.

The only fact about m_0 that is needed in [4] is that $a_{n+1} - a_n \in \{1, 2\}$ for all $n \geq m_0$. For the A and B sequences arising from special positions of End-Wythoff, this condition holds well before m_0, as the following proposition illustrates.

Proposition 1. *For all $n \geq r + 1$, $1 \leq a_{n+1} - a_n \leq 2$.*

Proof. $n \geq r+1$ implies that $b_{n+1} - b_n = a_{n+1} + n + 1 - a_n - n = a_{n+1} - a_n + 1 \geq 2$. That is, from index $r + 1$ onward, B contains no consecutive values. Therefore, since we know that $r + 1 \leq a_{r+1} \leq a_n$, (D) tells us that if $a_n + 1 \notin A$, then $a_n + 1 \in B$, so $a_n + 2 \notin B$, so $a_n + 2 \in A$, again by (D). This shows that $a_{n+1} - a_n \leq 2$. □

Therefore, in the first step of the initial computation, we are guaranteed to reach synchronization with $r + 1 \leq t \leq m$, where m is the index such that $a_m = b_{r+1} + 1$. Now, $a_{m_0} > b_r$ because $b_r \in V_{r+1} = X$. Also, $b_r \geq a_r$, so $a_{m_0} > a_r$ and $a_{m_0} \geq a_{r+1}$, which implies by (A) that $m_0 \geq r + 1$. Thus, this is an improvement over the bounds in the general case. Furthermore, note that as r grows larger, this shortcut becomes increasingly valuable.

5 Conclusion

We have exposed the structure of the P-positions of End-Wythoff, which is but a first study of this game. Many tasks remain to be done. For example, it would be useful to have an efficient method for computing l_K and r_K. The only method apparent from this analysis is unpleasantly recursive: if $K = (n_1, \ldots, n_k)$, then to find l_K, compute the P-positions for $(n_1, \ldots, n_{k-1})$ until reaching $(l_K, n_1, \ldots, n_{k-1}, n_k)$, and to find r_K, compute P-positions for $(n_2, \ldots, n_k)$ until reaching $(n_1, n_2, \ldots, n_k, r_K)$.

Additionally, there are two observations that one can quickly make if one studies special End-Wythoff positions for different values of r. Proving these conjectures would be a suitable continuation of this work:

- For $r \in \mathbb{Z}_{\geq 0}$, $\gamma = 0$.
- If $r \in \{0, 1\}$, then M, the number of subsequences of irregular shifts, equals 0. If $r = b'_n + 1$ and $a'_n + 1 \in B'$, then $M = 1$. Otherwise, $M = 3$.

Furthermore, evidence suggests that, with the appropriate bounds, Theorem 6 can be applied to any position of End-Wythoff rather than only special positions. In general, it seems that $b_n - a_n = n$ for $n > \max\{l_K, r_K\}$, if we enumerate only those P-positions with the leftmost pile smaller than or equal to the rightmost pile. This is another result that would be worth proving.

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