From Enmity to Amity

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1 Introduction

The On-line Encyclopedia of Integer Sequences [5] is well-known. It is of major assistance to numerous mathematicians and fuses together diverse lines of mathematical research. For example, searching for $2, 3, 4, 6, 9, 14, 22, 35, 56, 90, 145, \dots$ leads to sequence A001611, the "Fibonacci numbers + 1", listing about $\aleph_0$ comments, references, links, formulas, Maple and Mathematica programs and crossreferences to other sequences. Everybody can see that Sequence A000071, which lists the "Fibonacci numbers - 1", has even more material, so by the continuum hypothesis, it must contain at least $\aleph_1$ comments, references, links, formulas, Maple and Mathematica programs and crossreferences to other sequences; and Extensions.

Though there are two (unpublished) links common to the two sequences, the respective lists of references of the two sequences have an empty intersection; even in the "Adjacent sequences", there is no acknowledgment of the other. Moreover, there is no crossreference from one sequence to the other. This is astonishing, bordering on the offensive, since both sequences stem from the same source, the Fibonacci numbers. Are they antagonistic to each other? Our purpose is to show that there should be no animosity between the two sequences; both coexist peacefully in some applications.

2 Kimberling's Theorem

Let $F_{-2} = 0$, $F_{-1} = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 0)$ be the Fibonacci sequence. (For technical reasons this indexing differs from the usual.) Let $a(n) = \lfloor n\tau \rfloor$, $b(n) = \lfloor n\tau^2 \rfloor$, where $\tau = (1 + \sqrt{5})/2$ denotes the golden section. We consider iterations of these sequences. An example of an iterated identity is $a(b(n)) = a(n) + b(n)$. It can be abbreviated as $ab = a + b$, where the suppressed variable n is assumed to range over all positive integers, unless otherwise specified. Consider the word $w = \ell_1\ell_2 \dots \ell_k$ of length k over the binary alphabet $\{a, b\}$, where the product means iteration (composition). The number m of occurrences of the letter b is the *weight* of w. Recently, Clark Kimberling [4] proved the following nice and elegant result:

Theorem I. *For $k \geq 2$, let $w = \ell_1\ell_2 \ldots \ell_k$ be any word over $\{a, b\}$ of length k and weight m. Then $w = F_{k+m-4}a + F_{k+m-3}b - c$, where $c = F_{k+m-1} - w(1) \geq 0$ is independent of n.*

Notice that in the theorem — where $w(1)$ is w evaluated at $n = 1$ — only the weight m appears, not the locations within w where the b-s appear. Their locations, however, obviously influence the behavior of w. This influence is hidden in the "constant" $c = c_{k,m,\ell}$, where ℓ (location) indicates the dependence on the locations of the occurrences of b in w.

Examples. (i) Consider the case $m = 0$. Theorem I gives directly $a^k = F_{k-4}a + F_{k-3}b - F_{k-1} + 1$, since $\lfloor \tau \rfloor = 1$, so $w(1) = \lfloor \tau \ldots \lfloor \tau \lfloor \tau \rfloor \rfloor \ldots \rfloor = 1$.
(ii) $m = 1$, $w = ba^{k-1}$. Then $w(1) = \lfloor \tau^2 \lfloor \tau \ldots \lfloor \tau \lfloor \tau \rfloor \rfloor \ldots \rfloor = 2$, since $\lfloor \tau^2 \rfloor = 2$. Hence $ba^{k-1} = F_{k-3}a + F_{k-2}b - F_k + 2$.
(iii) $m = 1$, $w = a^{k-1}b$. Then $w(1) = a^{k-1}b(1) = \lfloor \tau \ldots \lfloor \tau \lfloor \tau^2 \rfloor \rfloor \ldots \rfloor$. What's the value of of this expression? The answer is given in the next section.

3 An Application

Theorem 1. *For $k \geq 1$, $w(1) = a^{k-1}b(1) = F_{k-1} + 1$, thus $c_{k,1,\ell} = F_{k-2} - 1$, and $w = a^{k-1}b = F_{k-3}a + F_{k-2}b - (F_{k-2} - 1)$.*

We see, in particular, that in a single theorem we have both "Fibonacci numbers + 1" (for $w(1)$) and "Fibonacci numbers - 1" (for $w = w(n)$), coexisting amicably.

Proof. We note that $b(1) = \lfloor \tau^2 \rfloor = 2 = F_0 + 1$, $ab(1) = \lfloor \tau \lfloor \tau^2 \rfloor \rfloor = \lfloor 2\tau \rfloor = 3 = F_1 + 1$, $a^2b(1) = \lfloor 3\tau \rfloor = 4 = F_2 + 1$. Suppose that $a^{k-1}b(1) = F_{k-1} + 1$ ($k \geq 3$). We split this assumption into two cases:

(i) $a^{2k}b(1) = F_{2k} + 1$ ($k \geq 1$), and (ii) $a^{2k+1}b(1) = F_{2k+1} + 1$ ($k \geq 1$).

The ratios F_k/F_{k-1} are the convergents of the simple continued fraction expansion of $\tau = [1, 1, 1, \ldots]$. Therefore $0 < \tau F_{2k+1} - F_{2k+2} < F_{2k+1}^{-1}$ and $-F_{2k}^{-1} < \tau F_{2k} - F_{2k+1} < 0$ (see e.g., [3], ch. 10). We may thus write,

$$\tau F_{2k+1} - F_{2k+2} = \delta_1, \text{ where } 0 < \delta_1 = \delta_1(k) < F_{2k+1}^{-1},$$

and

$$\tau F_{2k} - F_{2k+1} = \delta_2, \text{ where } -F_{2k}^{-1} < \delta_2 = \delta_2(k) < 0.$$

Using the induction hypothesis (i) we get,

$$a^{2k+1}b(1) = a(a^{2k}b(1)) = \lfloor \tau(F_{2k} + 1) \rfloor = F_{2k+1} + 1 + \lfloor \tau^{-1} + \delta_2 \rfloor = F_{2k+1} + 1,$$

since for $k \geq 1$, $F_{2k} \geq F_2 = 3$ so $-1/3 \leq \delta_2 < 0$, and $0.6 < \tau - 1 = \tau^{-1} < 0.62$.

Using (ii) we get similarly,

$$a^{2k+2}b(1) = a(a^{2k+1}b(1)) = \lfloor \tau(F_{2k+1} + 1) \rfloor = F_{2k+2} + 1 + \lfloor \tau^{-1} + \delta_1 \rfloor = F_{2k+2} + 1,$$

since for $k \geq 1$, $F_{2k+1} \geq 5$, so $0 < \delta_1 < 1/5$. ■

References

[1] A.S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly* **89** (1982) 353–361.

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[3] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Oxford, UK, 1960.

[4] C. Kimberling, Complementary equations and Wythoff sequences, *J. Integer Seq.* **11** (2008) 5 pp. (electronic).

[5] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at www.research.att.com/~njas/sequences/.

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