Martin Gardner and Wythoff's Game

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What's a question to your answer? We will not settle this puzzle here, yet we'll taste it. But let's begin at the beginning, namely in 1907, when Willem Abraham Wythoff [7], a Dutch mathematician, invented the game we analyze here, explained vividly by Martin Gardner in [5].

The game is played with two piles of tokens, each pile containing an arbitrary number of tokens. A move consists of either
(i) taking any positive number of tokens from a single pile, possibly the entire pile or
(ii) taking the same positive number of tokens from both piles.
The player who takes the last token wins. If both players have the same positive number of tokens, the next player wins by taking both piles. It's called an N-position, because the NEXT player wins. But if both piles are empty, the next player loses, and the previous player, the one who reached the empty piles wins. It's a P-position, because the PREVIOUS player wins.

It's easy to see that the positions $(0,1)$ and $(1,1)$ are N-positions, since the next player can win in one move. But $(1,2)$ is a P-position, since all its *followers* – positions reached in one move from a position – are N-positions. The first few P-positions are listed in Table 1. Note that every N-position has at least one P-follower. From an N-position a player will move to a P-position in order to win. The order of the two numbers in a pair (A_n, B_n) is unimportant, but throughout we arrange them in the order $0 \le A_n \le B_n$.

Table 1: The first few P-positions (A_n, B_n) for Wythoff's game.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
A_n	0	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25	27	29	30	32	33	35	37	38	40	42	43
B_n	0	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41	44	47	49	52	54	57	60	62	65	68	70

These two sequences, each strictly increasing, have remarkable properties. Note that $B_n = A_n + n$ for all n. But how is A_n computed? A study of Table 1 reveals that for $n \geq 1$, A_n is the smallest positive integer that has not yet appeared before in the two sequences. Thus, the next two entries in Table 1 are $A_{28} = 45$, $B_{28} = 73$. The latter property reveals that the two sequences are *complementary*: every positive integer appears precisely once in precisely one of the two sequences.

A little later we shall prove that the set of pairs $\{(A_n, B_n)\}$ constitutes the set of P-positions of the game. First note that we have constructed the sequences recursively. From a computational viewpoint, the implied winning strategy seems to be inefficient, because it appears that given any position (x, y) with $0 \leq x \leq y$ we have to construct Table 1 up to the smallest n such that $A_n \geq x$. Is there a nonrecursive way to generate them? Yes. Wythoff discovered that $A_n = \lfloor n\tau \rfloor$, $B_n = \lfloor n\tau^2 \rfloor$, where $\tau = (1 + \sqrt{5})/2$ is the golden ratio, and $\lfloor x \rfloor$ is the integer part of x, that is, x rounded down to the next integer. For example, $\lfloor 2 \rfloor = \lfloor 2.1 \rfloor = \lfloor 2.9 \rfloor = 2$. For every real x, $x - 1 < \lfloor x \rfloor \leq x$. The quadratic equation $x^2 - x - 1 = 0$ has the positive root $x = \tau$, hence $\tau^2 = \tau + 1$, leading again to $B_n = \lfloor n(\tau + 1) \rfloor = \lfloor n\tau \rfloor + n = A_n + n$.

The τ formulas provide a more efficient strategy computation than the recursive one. But sometimes, especially if we play with a very large number of tokens, it may be a bit hard to use them. Is there a simpler way to characterize the P-positions? Yes, there is, and it uses an exotic numeration system called the *Fibonacci* numeration system. The basis elements of the Fibonacci numeration system are the Fibonacci numbers F_n: $1, 2, 3, 5, 8, 13, 21, 34, \ldots$, where each number after the first two is the sum of its two predecessors. They replace the powers of 10 basis elements in the decimal system or powers of 2 in binary. We express a number such as 19 by taking the largest Fibonacci number not exceeding 19, namely 13, then doing the same for the difference $19 - 13 = 6$ which gives 5, repeating, which gives 1. Thus $19 = 1 \times 13 + 1 \times 5 + 1 \times 1$ and $50 = 1 \times 34 + 1 \times 13 + 1 \times 3$, see Table 2.

Table 2: The Fibonacci numeration system.

34	21	13	8	5	3	2	1	n
0	0	1	0	1	0	0	1	19
1	0	1	0	0	1	0	0	50

It's a binary numeration system (digits restricted to 0 and 1), with the property that there are no adjacent 1s, since two adjacent 1s produce the next higher Fibonacci number. Table 3 depicts the representation of the first few positive integers in the Fibonacci system. Compare it with Table 1. Can you tell which representations are A_n numbers and which are B_n numbers? We suggest that you concentrate attention at the right ends of the representations.

If you do that you might note that the representations of all A_n numbers end in an even number (possibly zero) of zeros; and so by complementarity, the representations of all B_n numbers end in an odd number of zeros. Moreover, taking a representation of any A_n and inserting a 0 at its right end (a *left shift* of A_n), yields B_n. Thus, shifting left 1 gives 10, and $(1, 10)$ is the representation of $(A_1, B_1) = (1, 2)$. Shifting left 100 results in 1000, and $(100, 1000)$ is the representation of $(A_2, B_2) = (3, 5)$.

Given any position (x, y) in Wythoff's game, we convert x and y into their Fibonacci number representations, which is a tractable computation. This enables us to verify whether or not (x, y) is a P-position. If it's an N-position, we can then move to a P-position. (Exercise: How is the move $P \to N$ done?) If it's a P-position, we can take just one token from a pile, a delaying tactic, hoping that our opponent will make an erroneous move.

Table 3: Representation of the integers $1 - 20$ in the Fibonacci system.

13	8	5	3	2	1	n
0	0	0	0	0	1	1
0	0	0	0	1	0	2
0	0	0	1	0	0	3
0	0	0	1	0	1	4
0	0	1	0	0	0	5
0	0	1	0	0	1	6
0	0	1	0	1	0	7
0	1	0	0	0	0	8
0	1	0	0	0	1	9
0	1	0	0	1	0	10
0	1	0	1	0	0	11
0	1	0	1	0	1	12
1	0	0	0	0	0	13
1	0	0	0	0	1	14
1	0	0	0	1	0	15
1	0	0	1	0	0	16
1	0	0	1	0	1	17
1	0	1	0	0	0	18
1	0	1	0	0	1	19
1	0	1	0	1	0	20

We are now ready to study the *generalized Wythoff game*. We wish to relax the condition of taking the *same* number of tokens from both piles, by permitting some perturbation. Specifically, in the generalized Wythoff game we can take any positive number of tokens from a single pile as in the original game, but we permit to take, say k from one pile and ℓ from the other, provided that $|k - \ell| < t$, where t is a fixed integer parameter. Note that $t = 1$ corresponds to the original game.

Consider the case $t = 2$. Then $(0,1)$, $(1,1)$ and $(1,2)$ are clearly N-positions, but $(1,3)$ is a P-position. The first few P-positions are displayed in Table 4.

Table 4: The first few P-positions (A_n, B_n) for $t = 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
A_n	0	1	2	4	5	7	8	9	11	12	14	15	16	18	19	21	22	24	25	26	28	29	31	32	33	35	36	38
B_n	0	3	6	10	13	17	20	23	27	30	34	37	40	44	47	51	54	58	61	64	68	71	75	78	81	85	88	92

Observe that now $B_n = A_n + 2n$, ($B_n = A_n + tn$ in general), and for $n \geq 1$, A_n is again the smallest positive integer that did not appear earlier in the two sequences. The sequences are therefore again complementary.

It's now time to prove that the set of pairs $\{(A_n, B_n)\}$ constitutes the set of P-positions. The proof is valid, in particular, for the original Wythoff game, which is simply the case $t = 1$. It evidently suffices to show two things: I. Every move from any position (A_n, B_n) results in a position not of the form (A_i, B_i). II. From any position $(x, y) \neq (A_i, B_i)$ there is a move to some (A_n, B_n).

I. Since the sequences are strictly increasing, a move of type (i) from (A_n, B_n) clearly results in a position not of the form (A_i, B_i). Suppose that a move $(A_n - k, B_n - k)$ of type (ii) is a position (A_i, B_i). Then $i < n$, yet $(B_n - k) - (A_n - k) = n$, a contradiction.

II. Let (x, y) with $0 \leq x \leq y$ be a position not of the form (A_i, B_i). Since $\{A_n\}$, $\{B_n\}$ are complementary, either $x = B_n$ or $x = A_n$ for some $n \geq 0$.

Case (i). $x = B_n$. Then move $y \to A_n$.

Case (ii). $x = A_n$. If $y > B_n$, then move $y \to B_n$. If $A_n \leq y < B_n$, let $d = y - x$, $m = \lfloor d/t \rfloor$, so $d/t - 1 < m \leq d/t$ and $d - tm \geq 0$. Then move $(x, y) \to (A_m, B_m)$. This is a legal move, since:

- $d = y - A_n < B_n - A_n = tn$, hence $m = \lfloor d/t \rfloor \leq d/t < n$.
- $y = A_n + d > A_m + d \geq A_m + tm = B_m$.
- $|(y - B_m) - (x - A_m)| = |(y - x) - (B_m - A_m)| = |d - tm| = d - tm < t$.

We now seek another "τ description" for the P-positions. More precisely, we seek two real numbers $\alpha > 0$ and $\beta > 0$ so that the sequences $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$ are identical to the sequences $\{A_n\}$ and $\{B_n\}$ respectively. In particular, they are complementary sequences.

Enter, the Canadian mathematician Samuel Beatty. He proved that if $\alpha > 0$ is irrational and $\alpha^{-1} + \beta^{-1} = 1$, then the sequences $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$

are complementary [1]. This does not hold if α is rational. The condition $\alpha^{-1}+\beta^{-1}=1$ is a natural *density* condition. For example, the sequence of even numbers and the sequence of odd numbers are complementary, that is, $\{2n\}$ and $\{2n-1\}$ $(n \ge 1)$, and $1/2+1/2=1$. Also the sequences $\{2n\}$, $\{4n-1\}$, $\{4n-3\}$ $(n \ge 1)$ are complementary and $1/2+1/4+1/4=1$. Actually, Beatty's discovery was predated by the British Nobel laureate (Physics) Lord Rayleigh [6] but people did not notice it.

So now we simply solve the quadratic equation $x^{-1}+(x+t)^{-1}=1$, which gives, for the positive root, $x=\alpha=(2-t+\sqrt{t^2+4})/2$, $\beta=\alpha+t$. For $t=1$ we recover our old friends $\alpha=\tau$, $\beta=\tau+1=\tau^2$; and for $t=2$, $\alpha=\sqrt{2}$, $\beta=\alpha+2$. Then for any positive integer t, the P-positions are the pairs $\{(\lfloor n\alpha\rfloor, \lfloor n\beta\rfloor)\}$. (Exercise: Prove this statement. Hint: Both the recursive and the explicit P-position versions consist of two complementary sequences.)

A strategy based on this observation can be realized as follows. Since α is irrational and $1<\alpha<2$,

$$x=\lfloor n\alpha\rfloor \iff x<n\alpha<x+1 \iff \frac{x}{\alpha}<n<\frac{x+1}{\alpha} \iff \left\lfloor\frac{x+1}{\alpha}\right\rfloor=\left\lfloor\frac{x}{\alpha}\right\rfloor+1,$$

where (x,y) with $0 \le x \le y$ is a given game position. Therefore either $x=\lfloor n\alpha\rfloor=A_n$, where $n=\lfloor (x+1)/\alpha\rfloor$, or else, by complementarity, $x=\lfloor n\beta\rfloor=B_n$, where $n=\lfloor (x+1)/\beta\rfloor$. We have thus reduced the situation to that of cases (ii) and (i) in the proof of the recursive version of the P-positions, so the strategy presented there can be followed.

Is there also an exotic numeration system for every positive t that generalizes the nice characterization of the P-positions derived for the Fibonacci numeration system?

Enter continued fractions. Any real α can be expanded in a *simple continued fraction*:

$$\alpha=a_0+\cfrac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3\dots}}}\ ,$$

where a_0 is an integer, and for $i \ge 1$, the a_i are positive integers. Because of its unwieldy form, it is customary to write it in the form $[a_0,a_1,a_2,a_3\dots]$. For example, $11/6=\underline{1}+5/6$, $6/5=\underline{1}+1/5$, $5/1=\underline{5}$. Hence $11/6=[1,1,5]$. If α is irrational, then its continued fraction is infinite. As an example, let $\alpha=[1,t,t,t,\dots]$. Put $\gamma:=[t,t,t,\dots]=t+1/[t,t,t,\dots]=t+1/\gamma$. This gives the quadratic equation $\gamma^2-t\gamma-1=0$, with positive solution $\gamma=(t+\sqrt{t^2+4})/2$, and $\alpha=[1,\gamma]=1+1/\gamma=(2-t+\sqrt{t^2+4})/2$, another old friend.

With the continued fraction of α we associate its *convergents* $p_n/q_n=[a_0,a_1,\dots,a_n]$, which are rational approximations of α. The convergents satisfy the recurrences $p_0=a_0$, $p_1=a_0a_1+1$, $p_n=a_np_{n-1}+p_{n-2}$ $(n \ge 2)$, $q_0=1$, $q_1=a_1$, $q_n=a_nq_{n-1}+q_{n-2}$ $(n \ge 2)$.

The numerators p_i of the convergents induce a numeration system in which every positive integer has a unique representation $N=\sum_{n\ge 0} d_np_n$, where the digits d_i satisfy $0 \le d_i \le a_{i+1}$, and $d_{i+1}=a_{i+2} \implies d_i=0$ $(i \ge 0)$. See

[4] for this, the generalized Wythoff game, a short proof of Beatty's Theorem and the basics about continued fractions. For $\alpha = [1, t, t, t, \ldots]$ we get, $p_0 = 1$, $p_1 = t + 1$, $p_n = tp_{n-1} + p_{n-2}$ $(n \geq 2)$, $q_0 = 1$, $q_1 = t$, $q_n = tq_{n-1} + q_{n-2}$ $(n \geq 2)$. The digits satisfy $0 \leq d_i \leq t$, $d_{i+1} = t \implies d_i = 0 \quad (i \geq 0)$. Note that for $t = 1$ the sequence of p_i is the Fibonacci sequence, and the numeration system is the Fibonacci numeration system. For $t = 2$, the representation of the numbers $1 - 20$ is displayed in Table 5.

Table 5: Representation of the integers $1 - 20$ for $t = 2$.

17	7	3	1	n
0	0	0	1	1
0	0	1	2	2
0	0	1	0	3
0	0	1	1	4
0	0	1	2	5
0	0	2	0	6
0	1	0	0	7
0	1	0	1	8
0	1	0	2	9
0	1	1	0	10
0	1	1	1	11
0	1	1	2	12
0	1	2	0	13
0	2	0	0	14
0	2	0	1	15
0	2	0	2	16
1	0	0	0	17
1	0	0	1	18
1	0	0	2	19
1	0	1	0	20

It's a ternary numeration system (digits restricted to 0, 1 and 2), with the property that if a digit is 2, then its right-hand neighbor is 0. This is a generalization of the rule in the Fibonacci system that there are no adjacent 1s. Exactly as in the Fibonacci system, also here the representations of all A_n numbers end in an even number (possibly zero) of zeros; and the representations of all B_n numbers end in an odd number of zeros. Moreover, taking a representation of any A_n and inserting a 0 at its right end yields B_n. Thus, shifting left 12 gives 120, and $(12, 120)$ is the representation of $(A_4, B_4) = (5, 13)$.

The connection between generalized Wythoff games and continued fractions, suggests to me the following "inverse problem". Suppose that we fix an arbitrary irrational $\alpha = [1, a_1, a_2, \ldots]$. Then $1 < \alpha < 2$. Let $\beta = \alpha/(\alpha - 1)$. Then $\alpha^{-1} + \beta^{-1} = 1$. Construct the sequences $\{(A_n, B_n)\} = \{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)\}$, which are necessarily complementary, and define them to be the P-positions of a game Γ. Formulate the game rules of Γ. Two colleagues – to be named once the blind refereeing is done – and I are currently investigating such inverse problems. Before that I have investigated such an inverse problem for the case $\alpha^{-1}+\beta^{-1} = p/q$, where p, q are positive integers in lowest terms, $q \geq 2$. Then $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$ are not complementary. This connects the end of our story with its beginning: we have the answer (the P-positions) and seek the question (the game rules).

Further study. Yet another characterization of the P-positions of Wythoff's game is based on the so-called *Wythoff word*, which plays a key-role in the theory of *combinatorics on words*. Readers interested in this approach may consult [2], [3]. As an exercise, they might wish to adapt this approach to generalized Wythoff games, by constructing a suitably generalized Wythoff word.

References

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[6] J. W. S. Rayleigh [1937], *The Theory of Sound*, MacMillan, London, 1st ed. 1877, 2nd ed. 1894, reprinted 1928,1929,1937. There is also a Dover 1945 publication, which is a reprint of the 2nd. The "sound" equivalent of Beatty's theorem, is stated, without proof, in sect. 92a, pp. 122-123, in the 1937 reprint of the 2nd ed., the only one I saw. I'm told that the 2nd edition contains the statement, but the 1st does not., MR0016009 (7,500e).

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