Solomon W. Golomb's Enlightening Games

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We thought that paying true tribute to the memory of Professor Solomon W. Golomb, would be to highlight a gem contribution of his on two-player games [1], which has not received the attention and respect it deserves, even among game experts. In this paper Sol analyzed two-player *take-away* games. We hope that this discourse will do justice to commemorating Sol's fruitful activities in a little-known direction, and, concurrently, illuminating an important yet neglected games corner.

A very simple take-away game is *Nim*. Given a number of positive integers, say 1, 2, 5, 13, the two players alternate in choosing one of the integers and reducing it, but so that the remaining integer is still nonnegative. The player who first reduced all integers to 0 is called the *winner*, the opponent the *loser*. There is a well-known winning strategy: express each integer in binary, and add them up in binary, but without carry. For the above case,

$$\begin{array}{cccc|l} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 1 & 13 \\ \hline 1 & 0 & 1 & 1 & 11 \end{array}$$

Alice wants to make this *Nim-sum* 0, by reducing a single row. In this case $13 \to 6$ will do the trick. However Bob now moves (reducing a single row), the resulting Nim-sum will be nonzero, and Alice can always reduce it to 0 by reducing a single row, until eventually arriving at all columns 0. Alice won since she made the last move.

In this missive, we always adopt the convention that the player making the last move wins (*normal* play). There are always precisely two players.

Sol began with an even simpler game than Nim, yet succeeded in using it, sleight of hand, as a *catalyst* to generate a class of much more sophisticated games – formulating and solving them. But before formulating the simpler catalyst game, let's jump ahead and present two examples of Sol's achievements in this direction. This can be done without yet disclosing the catalyst.

Notation. Denote by $\mathcal{N}$ the winning positions of the $\mathcal{N}$ext player, the player who moves *from* the current position. Also denote by $\mathcal{P}$ the winning positions of the $\mathcal{P}$revious player, the player who moved *to* the current position.

Notice that any end position u is in $\mathcal{P}$, since the next player cannot move from u. Every position v that has u as a direct follower is in $\mathcal{N}$, though v may have other direct followers in $\mathcal{N}$ or $\mathcal{P}$. In general, a position w is in $\mathcal{N}$ if and only if it has at least *one* direct follower in $\mathcal{P}$, whereas $w \in \mathcal{P}$ if and only if *all* of its direct followers are in $\mathcal{N}$.

I. Let $V = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \dots,\}$ the set of primes, with 1 adjoined, be the permitted moves. Sol claims that then $\mathcal{P} = \{4n : n \geq 0\}$. Quite surprising! Usually for take-away games, the move-set is rather regular, such as all of $\mathbb{N}$, offsets of arithmetic sequences, subsets of residues mod T for some integer T, etc, because only for those a winning strategy was found – not the set of primes! Moreover, notice the seemingly unlikely connection between the set of multiples of 4 and the set of all primes.

Illustrating the present case, suppose all positions < 8 are labeled according to Sol's claim. We have to show that $8 \in \mathcal{P}$. It suffices to show that all direct followers of 8 are in $\mathcal{N}$. So suppose that Alice begins to move $8 \to 7$ by subtracting 1. Now $7 \in \mathcal{N}$, since Bob can counter $7 \to 0$, winning. Similarly, if Alice moves to 5, 3, by subtracting 3, 5, respectively, then Bob can counter by moving to 0, winning. But Alice can move to 6 (by subtracting 2), which we have to show is in $\mathcal{N}$. Indeed, Bob moves $6 - 2 = 4 \in \mathcal{P}$. This confirms that $8 \in \mathcal{P}$: Alice lost. Now $9, 10, 11$ are in $\mathcal{N}$ by subtracting $1, 2, 3$ respectively. We leave it to the reader to show $12 \in \mathcal{P}$.

II. Let $V = \{1, 2, 4, 8, 16, 32, 64, 128, \dots,\}$ the set of nonnegative powers of 2 be the permitted moves. Sol claims that then $\mathcal{P} = \{3n : n \geq 0\}$. Again rather unexpected!

Illustration. Suppose we already know that all positions < 11 behave according to Sol's claim. We have to show that $11 \in \mathcal{N}$. Indeed, Alice moves $11 \to 3 \in \mathcal{P}$ by subtracting 8. Bob can now only move to either 1 or 2, from each of which Alice can move to 0. Alice won. It is now easy to show $12 \in \mathcal{P}$, $13, 14 \in \mathcal{N}$, $15 \in \mathcal{P}$.

Sol derived these and other results by beginning with the following simple catalyst game, dubbed $\mathbf{G}_k$ $(k \geq 1)$.

Let $T_k = \{1, 2, 3, \ldots, k\}$, where the permitted moves are to diminish the initial position by any positive integer $\leq k$. It is easy to see that $\mathcal{P}(T_k) = \{(k+1)n : n \geq 0\}$.

For example, if $T_k = T_8$ and the initial position is 50, then Alice will move $50 \to 45 = 5 \times 9$. Hereafter, Alice can always maintain a multiple of 9, whereas Bob never can. So eventually Alice will reach $0 \times 9 = 0$, winning.

Sol formulated and proved the following theorem, which demonstrates the connection between the above two sample games and the catalyst $\mathbf{G}_k$.

Theorem Let $V \subset \mathbb{N}$, and $U_k := T_k \cup V$. Then $\mathcal{P}(U_k) = \mathcal{P}(T_k)$ if and only if V is disjoint from $\mathcal{P}(T_k)$.

For the above "Primes" game, V is the set of all primes (and 1), $T_k = T_3 = \{1, 2, 3\}$, so $\mathcal{P}(T_3) = \{4n : n \geq 0\}$, which is disjoint from V. Hence $\mathcal{P}(U_3) = \{4n : n \geq 0\}$. But $U_3 = V$, since $T_3 \subset V$. Thus $\mathcal{P}(V) = \{4n : n \geq 0\}$.

For the "2-powers" game V is the set of all nonnegative powers of 2, $T_k = T_2 = 1, 2$, so $\mathcal{P}(T_2) = \{3n : n \geq 0\}$, which is disjoint from V. Analogously to the primes case, we get $\mathcal{P} = \{3n : n \geq 0\}$. The paper [1] contains considerably more sophisticated theorems and examples.

Actually Sol considered *vector* take-away games, where both positions and moves maintain nonnegative integer components throughout. But most of his effort in [1] is devoted to the 1-dimensional case, or to a multi-dimensional game, where only one component can be reduced at every move (such as Nim).

We have neglected to do justice to a very important aspect of [1], namely *shift registers* [3]. Here Sol fused together his talents in both electrical engineering and math to construct $\mathcal{P}$ sets for take-away games. A shift register is a linear sequence of cells connected together electronically, finite or infinite, depending on the size of the set V of permissible moves. Engineers of computer hardware or communication equipment are well-familiar with finite shift registers. The cells typically contain binary bits (0 or 1) and at each time interval are shifted right by one cell. The right-most bit is lost, and a new bit enters on the left. There are "feedback-taps" on the cells that correspond to V, the set of all moves. The contents of these cells are fed into an appropriate array of gates, whose output determines the bit that enters on the left. This procedure constructs the set $\mathcal{P}$ of V for take-away games.

Incidentally, shift register sequences are used in a broad range of applications, particularly in random number generation, multiple access and polling techniques, secure and privacy communication systems, error detecting and correcting codes, and synchronization pattern generation, as well as in modern cryptographic systems. Many of these were discovered and disseminated by Sol.

We attempted to adhere to Sol's original notation, but replaced just a little of it by modern nomenclature for game concepts, when we felt it would be more comprehensible to current readers.

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Professor Solomon W. Golomb loved puzzles – one person games: **SOL**o games. The most famous of these are *polyominoes*, which are plane configurations made up of unit squares, joined together by full edge to edge contact. Sol has made them very ubiquitous [2] by analysing them and demonstrating their usefulness in many diverse fields; Martin Gardner popularized them in one of his Scientific American columns and Dr Google is full of them. Like many puzzles in recreational mathematics, polyominoes raise many combinatorial problems. The most basic is enumerating polyominoes of a given size. No formula has been found except for special classes of polyominoes. A number of estimates are known, and there are algorithms for calculating them. In statistical physics, the study of polyominoes and their higher-dimensional analogs (which are often referred to as lattice animals in this literature) is applied to problems in physics and chemistry. Polyominoes have been used as models of branched polymers and of percolation clusters.

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In his talks and writings, Sol succeeded in making just the right compromise between two competing goals: math rigor and reader-friendliness: His statements and proofs are clear-cut rigorous, but he had a natural feeling for when they were beyond the grasp of the average reader; he filled the gap with explicit comprehensible examples, that provided the intuition and demonstrated what was going on, thus contributing huge reader-friendliness.

I encountered Sol first when I had just metamorphosed from an electric engineering Phd to a math Phd student at UCLA, where, inter alia, Sol was teaching. His lucid, luminous lectures were important ingredients in providing me with enthusiastic appreciation for mathematics.

References

[1] S. W. Golomb, A Mathematical Investigation of Games of "Take-Away", *J. Combinatorial Theory* **1** (1966), 443–458.

[2] S. W. Golomb, *Polyominoes: Puzzles, Patterns, Problems, and Packings* - Revised and Expanded Second Edition, Paperback, 200 pages Published April 7th 1996 by Princeton University Press (first published September 6th, 1994).

[3] S. W. Golomb, Shift register sequences, World Scientific, 3rd Revised edition (March, 2017).