Iterates of Beatty sequences

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Abstract

Analys of:

1. A paper of Ron Graham, whose memory we commemorate in this volume.

2. Beatty sequences and their iterates, also prompted by that paper of Ron.

We are interested in families of *Beatty sequences*: $\{\lfloor n\alpha_i + \beta_i \rfloor\}_{i=1}^{m}$, where $\alpha_i > 0, \beta$ are reals, $n = 1, 2, \ldots$.

It is well-known (e.g., see [10], [5], [7]) that for α_1, α_2 positive irrationals satisfying $\alpha_1^{-1} + \alpha_2^{-1} = 1$, the sequences $\{\lfloor n\alpha_1 \rfloor\}, \{\lfloor n\alpha_2 \rfloor\}, n = 1, 2, \ldots$ are *complementary*, that is, every positive integer appears exactly once, in exactly one of the sequences.

This doesn't work for rational α_1, α_2, say $5/3, 5/2$, since $3 \times 5/3 = 2 \times 5/2 = 5$. However for the *inhomogeneous* case, where also *offsets* β_i are permitted, complementary sequences $\lfloor n\alpha_i + \beta_i \rfloor$, exist for any reals $\alpha_i > 0, \beta_i$. For example, $\lfloor 5n/3 \rfloor, \lfloor 5n/2 - 1/2 \rfloor$ are complementary, as is verified easily. Also for integers there are inhomogeneous complementary sequences, namely arithmetic sequences, such as $2n, 2n-1$; $2n, 4n-1, 4n-3$.

Families in which all the α_i are integers, are denoted *Exactly Covering Congruences* (ECC); others where the α_i are either integers, or non-integer rationals or irrationals, are termed *Exactly Covering Families* (ECF); but, when necessary, we will qualify: rational ECF, irrational ECF, etc.

We quote from Graham [6] (with slight changes of wording): "It is easily seen that if $\{\lfloor na_i + b_i \rfloor\}_{i=1}^{m}$ is an ECC, and so is $\{\lfloor na_i' + b_i' \rfloor\}_{i=1}^{m'}$, and if $\{\lfloor n\alpha_1 + \beta_1 \rfloor, \lfloor n\alpha_2 + \beta_2 \rfloor\}$ is an irrational ECF, then their composition

$$\left\{\bigcup_{i=1}^{m}\lfloor na_i\alpha_1+b_i\alpha_1+\beta_1\rfloor\right\}\cup\left\{\bigcup_{i=1}^{m'}\lfloor na_i'\alpha_2+b_i'\alpha_2+\beta_2\rfloor\right\} \tag{1}$$

is an ECF.

The following is then stated in [6].

Theorem 1. *Any ECF with α_1, α_2 irrational must be of the form (1).*

If we write (1) in the form

$$\left\{\bigcup_{i=1}^{m}\lfloor (na_i+b_i)\alpha_1+\beta_1\rfloor\right\}\cup\left\{\bigcup_{i=1}^{m'}\lfloor (na_i'+b_i')\alpha_2+\beta_2\rfloor\right\}, \tag{2}$$

it becomes perhaps clearer what is done: replacing the multiplier n of α_1 by the union $(na_i + b_i)$, which is an ECC. But there is no reason to restrict ourselves to ECC.

In fact, we show, contrariwise to Theorem 1,

Theorem 2. *If $\lfloor n\gamma_1+\delta_1\rfloor, \lfloor n\gamma_2+\delta_2\rfloor$ is an irrational ECF, then any ECF admits the form (3),*

$$\left\{\bigcup_{i=1}^{m}\lfloor\lfloor n\alpha_i+\beta_i\rfloor\gamma_1+\delta_1\rfloor\right\}\cup\left\{\bigcup_{i=1}^{m'}\lfloor\lfloor n\alpha_i'+\beta_i'\rfloor\gamma_2+\delta_2\rfloor\right\}, \tag{3}$$

where $\{\lfloor n\alpha_i+\beta_i\rfloor : 1\leqslant i\leqslant m\}$ is any ECF, and so is $\{\lfloor n\alpha_i'+\beta_i'\rfloor : 1\leqslant i\leqslant m'\}$ (not necessarily ECC), $m, m'\geqslant 2$, $n = 1, 2, \ldots$.

Proof. Since $\lfloor(n\gamma_1+\delta_1)\rfloor, \lfloor(n\gamma_2+\delta_2)\rfloor$ $n = 1, 2, \ldots$ is an ECF, every positive integer appears exactly once in this ECF. Also $\lfloor n\alpha_i+\beta_i\rfloor$, $1\leqslant i\leqslant m$ is an ECF, so it can replace the multiplier n of $\lfloor(n\gamma_1+\delta_1)\rfloor$. We have *fragmented* (=*partitioned*) the multiplier n into m mutually ECF. □

Notation 1. We may sometimes write 'fragment a Beatty sequence' instead of the more cumbersome 'fragment the multiplier n of a Beatty sequence'.

Example 1. Let $a\geqslant 1$ be any integer. By direct computation it is easily verified that the irrationals α_1, α_2 defined by the equation

$$\left\{\alpha_1=\frac{2-a+\sqrt{a^2+4}}{2}, \alpha_2=\alpha_1+a\right\}, \tag{4}$$

satisfy $\alpha_1^{-1} + \alpha_2^{-1} = 1$.

For $a = 1$, $\alpha_1 = (1 + \sqrt{5})/2, \alpha_2 = (3 + \sqrt{5})/2$.

For $a = 2$, $\alpha_3 = \sqrt{2}, \alpha_4 = \sqrt{2} + 2$.

We also consider the non-integer rational system with $\alpha_5 = 5/3, \alpha_6 = 5/2, \beta_6 = -1/2$.

Illustrate: fragment the sequence $\lfloor n\alpha_3 \rfloor = \lfloor n\sqrt{2} \rfloor$ by the complementary rational sequences $\lfloor n\alpha_5 \rfloor = \lfloor 5n/3 \rfloor, \lfloor n\alpha_6 \rfloor = \lfloor (5n-1)/2 \rfloor$; and fragment the sequence $\lfloor n\alpha_4 \rfloor = \lfloor n(\sqrt{2} + 2) \rfloor$ by the complementary irrational sequences $\lfloor n\alpha_1 \rfloor = \lfloor n(1 + \sqrt{5})/2 \rfloor, \lfloor n\alpha_2 \rfloor = \lfloor n(3 + \sqrt{5})/2 \rfloor$. In Table (1), the non-fragmented sequences are in the two rightmost columns.

n	$\lfloor \lfloor n\alpha_5 \rfloor \alpha_3 \rfloor$	$\lfloor \lfloor n\alpha_6 \rfloor \alpha_3 \rfloor$	$\lfloor \lfloor n\alpha_1 \rfloor \alpha_4 \rfloor$	$\lfloor \lfloor n\alpha_2 \rfloor \alpha_4 \rfloor$	$\lfloor n\alpha_3 \rfloor$	$\lfloor n\alpha_4 \rfloor$
1	1	2	3	6	1	3
2	4	5	10	17	2	6
3	7	9	13	23	4	10
4	8	12	20	34	5	13
5	11	16	27	44	7	17
6	14	19	30	51	8	20
7	15	24	37	61	9	23
8	18	26	40	68	11	27
9	21	31	47	78	12	30
10	22	33	54	88	14	34
11	25	38	58	95	15	37
12	28	41	64	105	16	40
13	29	45	71	116	18	44
14	32	48	75	122	19	47
15	35	52	81	133	21	51

Table 1: Fragmenting the irrational sequence with $\alpha_3 = \sqrt{2}$, by the rational ECF $\lfloor n\alpha_5 \rfloor = \lfloor 5n/3 \rfloor, \lfloor n\alpha_6 \rfloor = \lfloor 5n/2 - 1/2 \rfloor$; and the irrational sequence with $\alpha_4 = \sqrt{2} + 2$ by the irrational ECF $\lfloor n\alpha_1 \rfloor = \lfloor n(1 + \sqrt{5})/2 \rfloor, \lfloor n\alpha_2 \rfloor = \lfloor n(3 + \sqrt{5})/2 \rfloor$.

Observe, e.g., that all numbers in the $\lfloor n\alpha_3 \rfloor$ column are distributed among the 2 adjacent columns containing α_3.

We now study the behavior of complementary Beatty sequences and their iterates, clearly revealing their nature.

We begin with the sequence of the positive integers, $n = 1, 2, \ldots$. Next, this sequence is fragmented into two pieces, called *Beatty Iterates* or simply *iterates* for short – by two complementary Beatty sequences, T_1, T_2. The two iterates fit together perfectly, like a *jigsaw puzzle*, because T_1, T_2 are complementary. Example: the 2 rightmost irrational sequences $\lfloor n\alpha_3 \rfloor, \lfloor n\alpha_4 \rfloor$ in Table 1. Next we can fragment one of the 2 iterates into 2 additional perfectly fitting jigsaw iterates by 2 more complementary Beatty sequences, such as the iterate $\lfloor n\alpha_3 \rfloor$. We fragment it by the 2 rational complementary sequences $\lfloor n\alpha_5 \rfloor$, $\lfloor n\alpha_6 \rfloor$ in Table 1. We can also fragment the other iterate, say by an irrational ECF, which is done in Table 1. So we have now a jigsaw puzzle consisting of 6 perfectly fitting iterates. As we mentioned, each of them is called a *Beatty Iterate*, BI.

There is no reason to stop here. Take, for example, $a = 3$ in Equation 4. Then $\alpha_7 = (-1 + \sqrt{13})/2, \alpha_8 = (5 + \sqrt{13})/2$. We use them to further fragment $\lfloor n\alpha_3 \rfloor$.

n	$\lfloor \lfloor \lfloor n\alpha_7 \rfloor \alpha_5 \rfloor \alpha_3 \rfloor$	$\lfloor \lfloor \lfloor n\alpha_8 \rfloor \alpha_5 \rfloor \alpha_3 \rfloor$	$\lfloor \lfloor n\alpha_6 \rfloor \alpha_3 \rfloor$	$\lfloor n\alpha_7 \rfloor$	$\lfloor n\alpha_8 \rfloor$	$\lfloor n\alpha_3 \rfloor$	$\lfloor n\alpha_4 \rfloor$
1	1	8	2	1	4	1	3
2	4	18	5	2	8	2	6
3	7	28	9	3	12	4	10
4	11	39	12	5	17	5	13
5	14	49	16	6	21	7	17
6	15	57	19	7	25	8	20
7	21	24	24	9	30	9	23
8	22	26	26	10	34	11	27
9	25	31	31	11	38	12	30
10	29	33	33	13	43	14	34
11	32	38	38	14	47	15	37
12	35	41	41	15	51	16	40
13	36	45	45	16	55	18	44
14	42	48	48	18	60	19	47
15	43	52	52	19	64	21	51

Table 2: Fragmenting the multiplier $\lfloor n\sqrt{5} \rfloor$ (see Table 1) of the irrational sequence with $\alpha_3 = \sqrt{2}$ by the irrational ECF $\lfloor n\alpha_7 \rfloor, \lfloor n\alpha_8 \rfloor$.

Thus we have produced a BI of degree $G = 3$ of $\lfloor n\sqrt{2} \rfloor$ of degree $G = 1$. In general we get iterates BIG of degree $G \geqslant 1$. A regular Beatty sequence is an iterate of degree $G = 1$. BI is independent from ECF. Of course we can join suitable BI into an ECF. For example in Table 2, adjoining $\lfloor n\alpha_4 \rfloor$ to the 3 columns containing α_3, results in an ECF.

We have proved:

Theorem 3. *Any BIG ($G \geqslant 1$) can be fragmented further into any BI of degree $\geqslant G$. Joining BI appropriately together produces ECF consisting of BI of arbitrary large degree.*

Thus, any ECF can produce other ECF by fragmentation!

We have fragmented ECF locally, and obtained bigger united ECF globally! The result is also more comprehensive, since we replaced ECC multipliers by general ECF multipliers.

In an attempt to salvage Theorem 1 of [6], perhaps every ECF fragmentation can also be fragmented by an ECC?

Now there are two ways of interpreting Theorem 1.

(1) Inclusive: Fragment by ECC but possibly also by ECF. We have shown that, in fact, it can always be fragmented into ECF. If this would be the interpretation of [6], then it would have to be stated in the form: Any ECF in which some α_i is irrational, can be fragmented by either ECC or ECF. [6]. Since this is not the case, the interpretation must be

(2) Exclusive: Fragment only by ECC, no other fragmentations are possible (which would anyway be the choice of "the man in the street"). But we have ECF counterexamples. Now the multiplier n of each counterexample can again be fragmented into ECC, thus possibly validating [6]. But they can also again be fragmented into ECF, thus violating exclusivity.

Thus, in any case, Theorem 1 appears to be invalid.

It is further stated in ([6]: "A theorem by Davenport, Mirsky and Newman asserts that in any ECC system, $m \geqslant 2$, the two largest α must be equal. This result and [6] then yielded the final

Corollary 1. *If* $\lfloor n\alpha_i + \beta_i \rfloor, 1 \leqslant i \leqslant m, m \geqslant 3$ *and the* α_i *irrational, then* $\alpha_i = \alpha_j$ *for some* $i \neq j$.

But the seeming invalidity of Theorem 1 implies that the corollary has not been proved. The corollary always seemed to provide a boost to FC. It is quite conceivable, however, that the corollary's *statement* is valid, since it is consistent with FC.

Epilogue Regarding this missive, I wrote to Ron Graham the following letter on June 26.

From: Aviezri S Fraenkel
Sent: Friday, June 26, 2020 6:24 AM
To: graham@ucsd.edu
Subject: ECF

Dear Ron,

How are you and yours doing during these pandemic times? Still joggling with $\aleph_2$ balls?

Re your slick 1973 Note, "Covering the positive integers ...", your main result is that any irrational ECF must be of the form (1) in your Note, i.e., fragmenting (= partitioning) a 2-term irrational ECF by ECC (exact covering congruences).

It seems to me, we can fragment the multiplying integer n of an ECF sequence S by any ECF (ECC, EFC rational or irrational), which induces a fragmentation of S.

The unnumbered formula in the attached missive displays a generic irrational ECF (depending on a): alpha and beta are obviously irrational in the formula, and they satisfy 1/alpha+1/beta=1.

In Table 1, the irrational ECF is the case $a = 2$ of the formula: The part $\sqrt{2}$ is fragmented by the rational ECF $5n/3, (5n-1)/2$. The part $\sqrt{2}+2$ is fragmented by case 1 of the formula.

In Table 2, an irrational ECF (case 2) is fragmented by another irrational ECF (case 1). The β_2 is not fragmented.

If the above is correct, then the theorem and climaxing final corollary of the Note are invalid. My hunch is they can be salvaged in the extended form. If so, then α_i in both can also be rational, even when the rational ECF has distinct moduli α_i as in my 1973 paper (which prompted your Note, I believe).

Perhaps you can show me easily that I get Fraenkel # minus infty, if all I wrote is nonsense. Or should I once more get Fraenkel # 0? Or Graham # 1? If the middle option, I may not have the energy to muster the proofs, but will write up an extended version of the attached missive.

Pls keep safe and healthy! Cheers, Aviezri.

Very sadly, Ron left us 10 days later, on July 6.

FC stands for Fraenkel conjecture (e.g., [3], [9]), stating that there is only a certain unique ECF with distinct α_i (which turns out to be a rational ECF).

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