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Variants of (s, t)-Wythoff's game[☆]

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ABSTRACT

In this paper, we study four games, they are all restrictions of (s, t)-Wythoff's game which was introduced by A.S. Fraenkel. The first one is a modular type restriction of (s, t)-Wythoff's game, where a player is restricted to remove a multiple of K tokens in each move (K is a fixed positive integer). The others we called rook type restrictions of (s, t)-Wythoff's game, including Odd-Arbitrary-Nim (s, t)-Wythoff's Game, Odd–Odd-Nim (s, t)-Wythoff's Game and Odd–Even-Nim (s, t)-Wythoff's Game. In these three games, the restrictions are only made on horizontal and vertical moves, but not on the extended diagonal moves. For any $K, s, t \geq 1$, the sets of P-positions of our games are given in both normal and misère play.

1. Introduction

Introduced by A.S. Fraenkel in [6], (s, t)-Wythoff's game is a well-known 2-player combinatorial game involving two piles of finitely many tokens. Given two integers $s, t \geq 1$, a player may either remove any positive number of tokens from a single pile or remove tokens from both piles, $k > 0$ from one pile and $\ell > 0$ from the other, say $\ell \geq k$, constrained by

$$0 \leq \ell - k < (s-1)k + t. \tag{1}$$

In normal play, the player first unable to move loses; while in misère play that player wins.

The special case $s = t = 1$ is the classical Wythoff game, while the case $s = 1, t \geq 1$ is Generalized Wythoff [4]. More variants of Wythoff's game and (s, t)-Wythoff's game can be found in [2,3,11,12,14,15]. For more theory of general combinatorial games, see [1,7,8,10].

By (a, b) we denote a game position with the two piles of sizes a and b. A position is called an N-position (known as winning position) from which the *Next* player can win. Otherwise, it is a P-position (known as losing position) from which the *Previous* player has a winning strategy. We denote by $\mathscr{P}$ and $\mathscr{N}$ the set of all P-positions of a game and the set of all its N-positions respectively. By $\mathbb{Z}^0$ and $\mathbb{Z}^+$ we denote the set of nonnegative integers and positive integers respectively.

Given any game, we notice that the set of all its P-positions constitutes an independent set, and the main goal is to find characterizations of the sequence of P positions. For example, in [6], the author gave all P-positions of (s, t)-Wythoff' game in normal play:

$$\mathscr{P} = \bigcup_{n=0}^{\infty} \{(A'_n, B'_n)\}, \quad A'_n = \text{mex}\,\{A'_i, B'_i \mid 0 \leq i < n\},\ B'_n = sA'_n + tn, \tag{2}$$

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where mex $S = \min(\mathbb{Z}^0 \setminus S)$. In particular, mex $\emptyset = 0$. In misère play, the set of all P-positions of (s, t)-Wythoff's game was determined in [13].

All four games in this paper are 2-player games played on two piles of finitely many tokens. Let

$$K \in \mathbb{Z}^+, \quad \mathcal{M}_K = \{nK \mid n \in \mathbb{Z}^0\}.$$

Now we define the first game which is a *modular type restriction* of (s, t)-Wythoff's game, denoted by Γ_K: Let $K, s, t \in \mathbb{Z}^+$, a player may either

I. remove k tokens, with $0 < k \in \mathcal{M}_K$, from a single pile, or
II. remove from both piles, k tokens from one pile with $0 < k \in \mathcal{M}_K$ and ℓ from the other with $0 < \ell \in \mathcal{M}_K$, subject to the constraint (1).

Notice that the case $K = 1$ is exactly (s, t)-Wythoff's game, while for $K = 2$, it is the "Even Even" case studied in [12].

The remaining three games are called Odd-Arbitrary-Nim (s, t)-Wythoff's game, Odd–Odd-Nim (s, t)-Wythoff's game and Odd–Even-Nim (s, t)-Wythoff's game. These games are *rook type restrictions* of (s, t)-Wythoff's game, which are denoted by Γ_{OA}, Γ_{OO}, Γ_{OE}, respectively. Throughout play of each of these three games, one pile is "first pile" and the other "second pile". In general, we denote by (x, y) a game position where x and y are the numbers of tokens in the first and the second pile, respectively.

(1) In Γ_{OA}, a player may either remove an *odd* number $k > 0$ of tokens from the first pile or an arbitrary number of tokens from the second pile, or move from both piles as in (s, t)-Wythoff's game.
(2) In Γ_{OO}, a player may only remove an *odd* number $k > 0$ of tokens when moving from a single pile (either the first or the second), while the move rule when moving from both piles is the same as that of (s, t)-Wythoff's game.
(3) In Γ_{OE}, a player may either remove an *odd* number $k > 0$ of tokens from the first pile or an *even* number $\ell > 0$ of tokens from the second, or move from both piles as in (s, t)-Wythoff's game.

Notice that in these three games no restriction is imposed on the diagonal move, while for Γ_K and the games defined in [12] also the diagonal move is constrained.

Section 2 provides methods for finding the P-positions of a game and its winning strategy. In Section 3, all P-positions of Γ_K are given recursively in terms of the mex function in both normal and misère play (Theorems 3 and 6). Moreover, a poly-time winning strategy for Γ_K in normal play is provided by exhibiting a relationship between Γ_K and (s, t)-Wythoff's game (Theorem 4 and Corollary 5), together with a special numeration system. While in misère play, a poly-time winning strategy for Γ_K is provided when $s = 1$ (Theorem 7 and Corollary 8). All P-positions of Γ_{OA}, Γ_{OO}, Γ_{OE} in both normal and misère play are given in Section 4 (Theorems 9–16), based on algebraic structures, which provide polynomial time strategies. The final Section 5 lists several far-reaching relevant open problems.

2. Preliminaries

It follows from the definition of P- and N-positions that from any N-position there always exists a move to a P-position and from a P-position a player can only move to an N-position (i.e., there can never be a move from a P-position to another P-position). These properties can be used to check whether a given position (a, b) is a P-position or not. By $F(u)$ we denote the *followers* of u, i.e., all positions that can be reached from u in one legal move. Symmetry of the game rules of Γ_K implies that both (a, b) and (b, a) are P-positions (or N-positions). For convenience, however, we agree to write (a, b) with $a \leq b$ throughout.

Example 1. For $K = s = 2$ and $t = 1$, consider Γ_K in normal play. We proceed according to the following steps to determine the first few P- and N-positions:

Step 1 P-positions: Clearly, $(0, 0), (0, 1), (1, 1) \in \mathscr{P}$, since the next player has no legal move from them and loses, that is, the previous player wins by default.

Step 2 N-positions: For $(0, m), (1, m), (m, m), (m, m+1), (m, m+2)$ with $m \geq 2$ and $(m, m+3)$ with m positive even, it is easy to check that from each of them a legal move of type I or II can result in a position in $\{(0, 0), (0, 1), (1, 1)\}$, thus they are all N-positions.

Step 3 P-positions: $F(2, 6) = \{(0, 2), (0, 4), (0, 6), (2, 2), (2, 4)\}$. It follows from Step 2 that each position of $F(2, 6)$ is an N-position. Thus $(2, 6) \in \mathscr{P}$. In the same manner, we can obtain that $(2, 7), (3, 6), (3, 7) \in \mathscr{P}$.

By repeating Steps 2 and 3, we can get more P-positions and N-positions of Γ_K.

3. Modular type restriction of (s, t)-Wythoff's game

We denote by $\lfloor x \rfloor$ the largest integer $\leq x$ and $\lceil x \rceil$ the smallest integer $\geq x$. By $\mathbb{Z}^{\geq m}$ we denote the set of all integers not less than m.

Definition 1. (i) For any set E and any element w, we define $E + w = \{e + w \mid e \in E\}$. In particular, $E = \emptyset \Longrightarrow E + w = \emptyset$.
(ii) Let $K, s, t \in \mathbb{Z}^+$, and $\Omega_K = \{0, 1, 2, \ldots, K-1\}$. We define two sequences A_n and B_n, for $n \in \mathbb{Z}^0$:

$$\begin{cases} A_n = \text{mex}\,\{\{A_i \mid 0 \le i < n\} + \alpha, \{B_i \mid 0 \le i < n\} + \beta\}, \quad \text{where } \alpha, \beta \in \Omega_K, \\ B_n = sA_n + \lceil t/K \rceil Kn. \end{cases} \tag{3}$$

Notice that for $K = 1$, $A_n = A'_n$, $B_n = B'_n$, where A'_n, B'_n were defined in Eq. (2).

Lemma 2. *Let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ be defined by Eq. (3). We have the following properties:*

(a) *$A_n, B_n \in \mathcal{M}_K$, for $n \in \mathbb{Z}^0$.*
(b) *For every m and n, with $n > m \ge 0$, we have $B_n > A_n > A_m$.*
(c) *Let $A = \bigcup_{n=1}^{\infty} \{A_n\} + \alpha$ and $B = \bigcup_{n=1}^{\infty} \{B_n\} + \beta$, where $\alpha, \beta \in \Omega_K$, with Ω_K being defined in Definition 1(ii). Then A and B are complementary with respect to $\mathbb{Z}^{\ge K}$, i.e., $A \cup B = \mathbb{Z}^{\ge K}$ and $A \cap B = \emptyset$.*
(d) *$A_n - A_{n-1} \in \{K, 2K\}$.*
(e) *$B_n - B_{n-1} \in \{sK + \lceil t/K \rceil K, 2sK + \lceil t/K \rceil K\}$. Moreover, $B_n - B_{n-1} = sK + \lceil t/K \rceil K$ if and only if $A_n - A_{n-1} = K$; $B_n - B_{n-1} = 2sK + \lceil t/K \rceil K$ if and only if $A_n - A_{n-1} = 2K$.*

Proof. (a) Induction on n. Obviously, $A_0 = B_0 = 0$, $A_1 = K$ and $B_1 = sA_1 + \lceil t/K \rceil K \in \mathcal{M}_K$. Suppose $A_j, B_j \in \mathcal{M}_K$ holds for all $j < n$. We now show that $A_n \in \mathcal{M}_K$, and so $B_n = sA_n + \lceil t/K \rceil Kn \in \mathcal{M}_K$.

Indeed, suppose that there exists some $q \in \mathbb{Z}^0$ such that $A_n = qK + \gamma$ with $0 < \gamma \in \Omega_K$. Let $S = \{\{A_i \mid 0 \le i < n\} + \alpha, \{B_i \mid 0 \le i < n\} + \beta\}$ with $\alpha, \beta \in \Omega_K$. Then we have $qK + \gamma = \text{mex}\, S$. This implies that $qK + \gamma \notin S$ and $qK = A_n - \gamma \in S$. If there exist $i_0 < n$ and $\alpha, \beta \in \Omega_K$ such that $qK = A_{i_0} + \alpha$ or $qK = B_{i_0} + \beta$, then by assumption $A_{i_0}, B_{i_0} \in \mathcal{M}_K$ implying that $\alpha = \beta = 0$. Hence $qK + \gamma = A_{i_0} + \gamma \in S$ or $qK + \gamma = B_{i_0} + \gamma \in S$, giving a contradiction.

(b) A_n and B_n are strictly increasing sequences, which is obvious from their definition, and $B_n = sA_n + \lceil t/K \rceil Kn \ge A_n + Kn > A_n > A_m$, for any $n > m \ge 0$.

(c) It is easy to see that $A \cup B = \mathbb{Z}^{\ge K}$. Suppose $A \cap B \ne \emptyset$. It follows from (a) that $A_m + \alpha' \ne B_n$ and $A_m \ne B_n + \beta'$ with $\alpha' > 0, \beta' > 0$, thus the only possibility is $A_m = B_n$ for some integers $m, n \in \mathbb{Z}^+$. If $m > n$, then A_m is mex of a set containing $B_n = A_m$, a contradiction. If $m \le n$, then by (b) we have $B_n = sA_n + \lceil t/K \rceil Kn \ge sA_m + \lceil t/K \rceil Km > A_m$, another contradiction.

(d) By (a) and (b), $0 < A_n - A_{n-1} \in \mathcal{M}_K$. Assume that $A_n - A_{n-1} \ge 3K$, then $A_{n-1} < A_{n-1} + K < A_{n-1} + 2K < A_{n-1} + 3K \le A_n$. By (c), $A_{n-1} + \omega \in S$ with $1 \le \omega \le 3K - 1$. Further, the only possibility is that $A_{n-1} + \omega \in B$. Since $A_n, B_n \in \mathcal{M}_K$, there exists some $j < n$ such that $A_{n-1} + K = B_j$ and $A_{n-1} + 2K = B_{j+1}$. Hence, we get $K = B_{j+1} - B_j = s(A_{j+1} - A_j) + \lceil t/K \rceil K > K$, a contradiction.

(e) Directly from the definition of B_n and (d). ∎

Theorem 3. *Let $K, s, t \in \mathbb{Z}^+$. For Γ_K in normal play,*

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \{(A_n + \alpha, B_n + \beta) \mid \alpha, \beta \in \Omega_K\},$$

where A_n and B_n are defined in Eq. (3) and Ω_K in Definition 1(ii).

Proof. It evidently suffices to show two things:

Fact I. (stability property). No followers of a position in $\mathcal{P}$ can be in $\mathcal{P}$.

Fact II. (absorbing property). From every position not in $\mathcal{P}$ there is a move to a position in $\mathcal{P}$.

Proof of Fact I. Let (x, y) with $x \le y$ be a position in $\mathcal{P}$. Clearly for $(x, y) \in \Omega_K \times \Omega_K$, with Ω_K being defined in Definition 1. For $x, y \ge K$, it follows from Lemma 2(c) that there exist some $n \in \mathbb{Z}^+$ and $\alpha, \beta \in \Omega_K$ such that $(x, y) = (A_n + \alpha, B_n + \beta)$.

It is obvious that a type I move from (x, y) leads to a position not in $\mathcal{P}$. Suppose that $(x, y) \to (x', y') \in \mathcal{P}$ by a type II move. By Lemma 2(a) and (b), there exists m $(< n)$ such that $k = A_n - A_m \in \mathcal{M}_K$ and $\ell = B_n - B_m \in \mathcal{M}_K$. Note that $\lceil t/K \rceil K \ge t$ for any $K, t \in \mathbb{Z}^+$, thus $0 < k \le \ell = s(A_n - A_m) + \lceil t/K \rceil K(n - m) \ge sk + t$, which contradicts Eq. (1).

Proof of Fact II. Let (x, y) with $x \le y$ be a position not in $\mathcal{P}$. If $x \in \Omega_K$, let $y = qK + \beta$, $q \in \mathbb{Z}^+$ and $\beta \in \Omega_K$, then move $y \to \beta$. If $x \ge K$, from Lemma 2(c), we have either $x = B_n + \beta$ or $x = A_n + \alpha$ for some $n \in \mathbb{Z}^+$ and $\alpha, \beta \in \Omega_K$.

Case (i) $x = B_n + \beta$. Let $y = qK + \alpha$, $q \in \mathbb{Z}^0$ and $\alpha \in \Omega_K$, we move $y \to A_n + \alpha$, since $y \ge x = B_n + \beta \ge B_n > A_n + \alpha$ and $y - A_n - \alpha \in \mathcal{M}_K$.

Case (ii) $x = A_n + \alpha$. In this case, let $y = qK + \beta$, $q \in \mathbb{Z}^0$, $\beta \in \Omega_K$. We proceed by distinguishing three subcases: (ii.1) $y > B_n + K - 1$, (ii.2) $x \le y < sA_n + \lceil t/K \rceil K$, (ii.3) $sA_n + \lceil t/K \rceil K \le y < B_n$.

(ii.1) $y > B_n + K - 1$. Then move $y \to B_n + \beta$.

(ii.2) $x \le y < sA_n + \lceil t/K \rceil K$. We move $(x, y) \to (\alpha, \beta)$. This move is legal: (a) $0 < k = A_n \in \mathcal{M}_K$, (b) $0 < \ell = y - \beta \in \mathcal{M}_K$, (c) $\ell - k = y - \beta - A_n \le (s-1)A_n + \lceil t/K \rceil K - K < (s-1)k + t$.

(ii.3) $sA_n + \lceil t/K \rceil K \le y < B_n$. Put $m = \lfloor (y - sA_n - \beta)/(\lceil t/K \rceil K) \rfloor$. Then move $(x, y) \to (A_m + \alpha, B_m + \beta)$. This move is legal:

Table 1
The first few P-generators of Γ_3.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A_n	0	3	6	9	15	18	21	27	30	33	39	42	45	48
B_n	0	12	24	36	54	66	78	96	108	120	138	150	162	174

Table 2
The first few P-positions of the associated Γ.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A'_n	0	1	2	3	5	6	7	9	10	11	13	14	15	16
B'_n	0	4	8	12	18	22	26	32	36	40	46	50	54	58

(a) $0 < k \in \mathcal{M}_K$. We first prove $0 \le m < n$. Since $y - sA_n \ge \lceil t/K \rceil K \ge K > \beta$, then $(y - sA_n - \beta)/(\lceil t/K \rceil K) > 0$, and so $m = \lfloor (y - sA_n - \beta)/(\lceil t/K \rceil K) \rfloor \ge 0$. On the other hand, $y - sA_n - \beta < B_n - sA_n = \lceil t/K \rceil Kn$, thus $m = \lfloor (y - sA_n - \beta)/(\lceil t/K \rceil K) \rfloor \le (y - sA_n - \beta)/(\lceil t/K \rceil K) < n$. Hence $k = A_n - A_m > 0$.

(b) $0 < \ell \in \mathcal{M}_K$. We know $m \le (y - sA_n - \beta)/(\lceil t/K \rceil K)$, it follows that $y \ge \lceil t/K \rceil Km + sA_n + \beta = B_m + \beta + s(A_n - A_m) > B_m + \beta$. Thus $\ell = y - B_m - \beta > 0$ and clearly $\ell \in \mathcal{M}_K$.

(c) $k \le \ell < sk + t$. By the definition of m, we have $m > (y - sA_n - \beta)/(\lceil t/K \rceil K) - 1$, then $y < \lceil t/K \rceil K(m + 1) + sA_n + \beta$. Thus $y - B_m - \beta < s(A_n - A_m) + \lceil t/K \rceil K$. Further, $y - B_m - \beta \le s(A_n - A_m) + \lceil t/K \rceil K - K < s(A_n - A_m) + t$. On the other hand, by (b), $y - B_m - \beta \ge s(A_n - A_m) \ge A_n - A_m$. ■

Theorem 3 provides a recursive winning strategy which is exponential in the input size $\log xy$ of any game position $(x, y) \in \mathbb{Z}^0 \times \mathbb{Z}^0$.

For every $n \in \mathbb{Z}^0$, the pair (A_n, B_n) is called a *P-generator* of P-positions, since the pair generates the set $\{(A_n + \alpha, B_n + \beta) \mid \alpha, \beta \in \Omega_K\}$ of P-positions, with Ω_K being defined in Definition 1(ii).

Now the original (s, t)-Wythoff's game with parameters $s, t \in \mathbb{Z}^+$ is the case $K = 1$ of Γ_K. Its P-positions are exactly those in Eq. (2). With $\Gamma_K, K > 1$, we *associate* an (s, t')-Wythoff game

$$\Gamma := \Gamma_1$$

with parameters $s(\Gamma) = s(\Gamma_K)$, $t'(\Gamma) = \lceil t/K \rceil$, K as in Γ_K.

In order to provide a poly-time winning strategy for Γ_K, we next exhibit a simple relationship between the P-generators of Γ_K and the P-positions of the associated Γ, which are those of (2), but with t replaced by t':

Theorem 4. *$A'_n = A_n/K$, $B'_n = B_n/K$, where $\{(A_n, B_n)\}_{n\ge 0}$ and $\{(A'_n, B'_n)\}_{n\ge 0}$ are the P-generators of Γ_K and the P-positions of Γ respectively.*

Example 2. For $K = 3$, $s = 2$, $t \in \{4, 5, 6\}$, we display the first few P-generators of Γ_3 and the first few P-positions of the associated Γ in Tables 1 and 2. Notice the divisibility enunciated by Theorem 4.

Proof. From Lemma 2, for all $n \ge 0$: (i) $A_n, B_n \in \mathcal{M}_K$, (ii) $A_{n+1} - A_n \in \{K, 2K\}$, (iii) $B_{n+1} - B_n \in \{sK + \lceil t/K \rceil K, 2sK + \lceil t/K \rceil K\}$.

We see, in particular, that A_n/K, B_n/K are nonnegative integers.

From the proof of Theorem 3.1 of [6] we have: (i)′ $A'_{n+1} - A'_n \in \{1, 2\}$, (ii)′ $B'_{n+1} - B'_n \in \{s + t', 2s + t'\}$.

(i)′, (ii)′ follow from (ii), (iii) respectively by dividing by K. But the theorem is not yet proved: it could presumably happen, for example, that for some $n \ge 0$, $A_{n+1} - A_n = 2K$, yet $A'_{n+1} - A'_n = 1$ rather than 2. We now show, however, by induction on n, that

$$(A_{n+1} - A_n)/K = A'_{n+1} - A'_n, \qquad (B_{n+1} - B_n)/K = B'_{n+1} - B'_n \tag{4}$$

for all $n \ge 0$. The theorem's assertion clearly holds for $n = 0$. Further, from the definition of A_n, B_n we get: $A_1 = K$, $B_1 = sK + \lceil t/K \rceil K$; and from the definition of A'_n, B'_n: $A'_1 = 1$, $B'_1 = s + t'$. Thus Eq. (4) holds for $n = 0$. Suppose $(A_{j+1} - A_j)/K = A'_{j+1} - A'_j$, $(B_{j+1} - B_j)/K = B'_{j+1} - B'_j$ hold for all $j < n$. If $A_{n+1} = A_n + K$, it follows from the mex function and the induction hypothesis that $A'_{n+1} = A'_n + 1$. Similarly, $A_{n+1} = A_n + 2K$ implies $A'_{n+1} = A'_n + 2$. Also B_{n+1}, B'_{n+1} are uniquely determined by A_{n+1}, A'_{n+1} respectively. Thus, again by the induction hypothesis (on A_n, A'_n), Eq. (4) is established, so the theorem's assertion follows. ■

Corollary 5. *In normal play, (x, y) is a P-position of Γ_K if and only if $(\lfloor x/K \rfloor, \lfloor y/K \rfloor)$ is a P-position of Γ.*

Proof. If (x, y) is a P-position of Γ_K with its P-generator being (A_{i_0}, B_{i_0}), $i_0 \in \mathbb{Z}^0$, then by Theorem 4, $(\lfloor x/K \rfloor, \lfloor y/K \rfloor) = (A'_{i_0}, B'_{i_0})$, and vice versa. ■

Table 3
Representations $R(N)$ over $\mathcal{U}$.

14	4	1	N
		1	1
		2	2
		3	3
	1	0	4
	1	1	5
	1	2	6
	1	3	7
	2	0	8
	2	1	9
	2	2	10
	2	3	11
	3	0	12
	3	1	13
1	0	0	14
1	0	1	15
1	0	2	16
1	0	3	17
1	1	0	18
1	1	1	19
1	1	2	20

We now show how Theorem 4 leads to a poly-time winning strategy for Γ_K. Let $u_{-1} = 1/s$, $u_0 = 1$, $u_n = (s + t' - 1)u_{n-1} + su_{n-2}$ $(n \geq 1)$. Denote by $\mathcal{U}$ the numeration system with bases $u_0, u_1, \ldots$ and digits $d_i \in \{0, \ldots, s + t' - 1\}$ such that $d_{i+1} = s + t' - 1 \Longrightarrow d_i < s$ $(i \geq 0)$. In [6] it was shown (as a special case of a somewhat more general numeration system) that every positive integer N has a unique representation $R(N)$ over $\mathcal{U}$.

The *vile* numbers are those whose representations $R(N)$ end in an even number of 0s, and the *dopey* numbers are those whose representations end in an odd number of 0s. (For an explanation/etymology of the terms vile, dopey, see [9].) Also, y is a *left shift* of x, if $R(y)$ is obtained from $R(x)$ by adjoining 0 to the right end of $R(x)$. Thus, in binary, the decimal number 6 is a left shift of the decimal 3, since $R(6) = 110$, $R(3) = 11$; 3 is vile since $R(3)$ ends in an even number (zero) of 0s and 6 is dopey.

In [6] it was proved that $(x, y) \in \Gamma$ with $x \leq y$ is a P-position of Γ if and only if x is vile and y is a left shift of x (so it is dopey). The fact that the u_i grow exponentially, together with Theorem 4 clearly provides a poly-time winning strategy for Γ_K. For $K = 2$ this provides a poly-time winning strategy for the "Even Even" case, which remained elusive in [12].

Notice that if s, t are the parameters of Γ_K, then s, t' are the parameters of Γ, where $t' = \lceil t/K \rceil$.

Example 3. Consider Γ_3 of Example 2, where $K = 3, s = 2, t \in \{4, 5, 6\}$. Then the corresponding game Γ has values $s = t' = 2$. Thus, $u_{-1} = 1/2, u_0 = 1, u_1 = 4, u_2 = 14, u_3 = 50, \ldots$. The representations $R(N)$ over $\mathcal{U}$ of the first few positive integers N appear in Table 3. Consider the position $(4, 17) \in \Gamma_3$. By Corollary 5, we check $(\lfloor 4/3 \rfloor, \lfloor 17/3 \rfloor) = (1, 5)$ and their representations $(1, 11)$. Since 11 is not a left shift of 1 (but 1 ends in an even number of 0s), $(1, 5)$ is an N-position in Γ, hence $(4, 17)$ is an N-position in Γ_3. Now consider $(11, 37) \in \Gamma_3$, so $(\lfloor 11/3 \rfloor, \lfloor 37/3 \rfloor) = (3, 12)$, with representations $(3, 30)$. Since 3 ends in an even number of 0s and 30 is a left shift of 3, $(3, 30)$ is a P-position in Γ, hence $(11, 37)$ is a P-position in Γ_3.

Theorem 6. *Let $K, s, t \in \mathbb{Z}^+$. For Γ_K in misère play, $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(E_n + \alpha, H_n + \beta) \mid \alpha, \beta \in \Omega_K\}$, where Ω_K is defined in Definition 1*(ii)*, E_n and H_n are determined by two cases:*

(A) *If $s > 1$ or $t > K$, then for $n \in \mathbb{Z}^0$,*

$$\begin{cases} E_n = \operatorname{mex}\{\{E_i \mid 0 \leq i < n\} + \alpha, \{H_i \mid 0 \leq i < n\} + \beta\}, \\ H_n = sE_n + \lceil t/K \rceil Kn + K. \end{cases} \tag{5}$$

(B) *If $s = 1$ and $t \leq K$, then $E_0 = H_0 = 2K$ and for $n \in \mathbb{Z}^+$,*

$$\begin{cases} E_n = \operatorname{mex}\{\{E_i \mid 0 \leq i < n\} + \alpha, \{H_i \mid 0 \leq i < n\} + \beta\}, \\ H_n = E_n + Kn. \end{cases} \tag{6}$$

Example 4. For $K = 3, s = 2, t \in \{4, 5, 6\}$, we display the first few P-generators of Γ_K in Table 4, which shows us how to determine $\mathscr{P}$ by using Eq. (5).

Proof. Let $E = \bigcup_{n=0}^{\infty}\{E_n\} + \alpha$ and $H = \bigcup_{n=0}^{\infty}\{H_n\} + \beta$ with $\alpha, \beta \in \Omega_K$. We firstly claim the following facts:

Fact A Suppose $s > 1$ or $t > K$.

I. Similar to Lemma 2(a) and (b), $E_n, H_n \in \mathcal{M}_K$ and it is easy to see that both E_n and H_n are strictly increasing sequences, for $n \in \mathbb{Z}^0$.

Table 4
The first few P-generators of Γ_K for $K = 3, s = 2, t \in \{4, 5, 6\}$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
E_n	0	6	9	12	15	18	24	27	30	36	39	42	48
H_n	3	21	33	45	57	69	87	99	111	129	141	153	171

II. $E \cup H = \mathbb{Z}^0$ and $E \cap H = \emptyset$. In fact, $E \cup H = \mathbb{Z}^0$ follows from the definition of mex. Now suppose $E \cap H \neq \emptyset$. It follows Fact A.I that $E_m + \alpha' \neq H_n$ and $E_m \neq H_n + \beta'$ with $\alpha' > 0, \beta' > 0$, thus the only possibility is $E_m = H_n$ for two integers $m, n \in \mathbb{Z}^+$. If $m > n$ then $E_m = \text{mex}\,\{E_i + \alpha, H_i + \beta \mid 0 \le i < m,\ \alpha, \beta \in \Omega_K\}$, which contradicts $E_m = H_n$; if $m \le n$ then $H_n \ge sE_m + \lceil t/K \rceil Kn + K > E_m$, also contradicting $E_m = H_n$.

Fact B Suppose $s = 1$ and $t \le K$.

I. $E_n, H_n \in \mathcal{M}_K$ for $n \in \mathbb{Z}^0$ and E_n, H_n are strictly increasing sequences for $n \in \mathbb{Z}^+$.
II. $E \cup H = \mathbb{Z}^0$ and $E \cap H = \{2K\}$. Its proof is similar to that of Fact A.II.

Proof of Fact I. Let (x, y) with $x \le y$ be a position in $\mathscr{P}$. There exist some $n \in \mathbb{Z}^0$ and $\alpha, \beta \in \Omega_K$ such that $(x, y) = (E_n + \alpha, H_n + \beta)$.

It is easy to check that no move of type I from (x, y) can terminate in $\mathscr{P}$. Then suppose $(x, y) \to (x', y') \in \mathscr{P}$ by a type II move, and there exists some m such that $(x', y') = (E_m + \alpha, H_m + \beta)$. Thus for both cases (A) and (B), we have $k = E_n - E_m > 0$, $\ell = H_n - H_m$ and $0 < k \le \ell = s(E_n - E_m) + \lceil t/K \rceil K(n - m) \ge sk + t$, which contradicts Eq. (1).

Proof of Fact II. Let (x, y) with $x \le y$ be a position not in $\mathscr{P}$. By Facts A.II and B.II, we have either $x = H_n + \beta$ or $x = E_n + \alpha$, for some $n \in \mathbb{Z}^0$ and $\alpha, \beta \in \Omega_K$.

Case (i) $x = H_n + \beta$. Now $y \ge E_n + K$. Let $y = qK + \alpha$, $q \in \mathbb{Z}^0$, and $\alpha \in \Omega_K$. Then move $y \to E_n + \alpha$, since $0 < y - E_n - \alpha \in \mathcal{M}_K$.

Case (ii) $x = E_n + \alpha$. In this case, we have $y > H_n + K - 1$ or $x \le y < H_n$. Let $y = qK + \beta$, where $q \in \mathbb{Z}^0$, and $\beta \in \Omega_K$. If $y > H_n + K - 1$, then move $y \to H_n + \beta$, since $0 < y - H_n - \beta \in \mathcal{M}_K$. If $x \le y < H_n$, we consider two subcases: (ii-A) $s > 1$ or $t > K$; (ii-B) $s = 1$ and $t \le K$.

(ii-A) $s > 1$ or $t > K$.

For $n = 0$, we have $x \le y < K = H_0$, the next player wins without doing anything.

For $n \ge 1$. If $x \le y < sE_n + \lceil t/K \rceil K + K$, move $(x, y) \to (\alpha, K + \beta)$. This is a legal move, since $k = E_n$, $\ell = y - K - \beta$, and $0 \le \ell - k < sE_n + \lceil t/K \rceil K - \beta - E_n \le (s-1)E_n + \lceil t/K \rceil K - K < (s-1)k + t$. If $sE_n + \lceil t/K \rceil K + K \le y < H_n$, put $m = \lfloor (y - sE_n - K - \beta)/(\lceil t/K \rceil K) \rfloor$ and move $(x, y) \to (E_m + \alpha, H_m + \beta)$. This move is legal:

(a) $0 < k \in \mathcal{M}_K$. Clearly $k = E_n - E_m \in \mathcal{M}_K$. It suffices to prove that $0 \le m < n$. Note that $y - sE_n - K \ge \lceil t/K \rceil K \ge K > \beta$, so $(y - sE_n - K - \beta)/(\lceil t/K \rceil K) > 0$, thus $m = \lfloor (y - sE_n - K - \beta)/(\lceil t/K \rceil K) \rfloor \ge 0$. On the other hand, $y - sE_n - K - \beta < H_n - sE_n - K = \lceil t/K \rceil Kn$, and so $m = \lfloor (y - sE_n - K - \beta)/(\lceil t/K \rceil K) \rfloor \le (y - sE_n - K - \beta)/(\lceil t/K \rceil K) < n$.

(b) $0 < \ell \in \mathcal{M}_K$. It is obvious that $\ell = y - H_m - \beta = qK - H_m \in \mathcal{M}_K$. Now $m \le (y - sE_n - K - \beta)/(\lceil t/K \rceil K)$, So $y \ge \lceil t/K \rceil Km + sE_n + K + \beta = H_m + \beta + s(E_n - E_m) > H_m + \beta$.

(c) $k \le \ell < sk + t$. By above, $m > (y - sE_n - K - \beta)/(\lceil t/K \rceil K) - 1$, i.e., $y < \lceil t/K \rceil K(m + 1) + sE_n + K + \beta$. So $y - H_m - \beta < s(E_n - E_m) + \lceil t/K \rceil K$, thus we have $\ell = y - H_m - \beta \le s(E_n - E_m) + \lceil t/K \rceil K - K < s(E_n - E_m) + t = sk + t$. On the other hand, by (b), $\ell = y - H_m - \beta \ge s(E_n - E_m) \ge E_n - E_m = k$.

(ii-B) $s = 1$ and $t \le K$.

If $n = 0$, then $2K + \alpha = x \le y < H_0 = 2K$ is impossible; if $n = 1$ then $0 \le x \le y \le K - 1$, thus the next player wins without doing anything. It remains to consider the case $n \ge 2$:

Put $m = \lfloor (y - E_n - \beta)/K \rfloor$ and move $(x, y) \to (E_m + \alpha, H_m + \beta)$. This move is legal: (a) $0 < k = E_n - E_m \in \mathcal{M}_K$. As above, we only need to prove that $0 \le m < n$. Since $y \ge E_n + \beta$, then $m = \lfloor (y - E_n - \beta)/K \rfloor \ge 0$. On the other hand, $y - E_n - \beta < H_n - E_n = Kn$, and so $m = \lfloor (y - E_n - \beta)/K \rfloor \le (y - E_n - \beta)/K < n$.

(b) $0 < \ell \in \mathcal{M}_K$. Obviously, $\ell = y - H_m - \beta = qK - H_m \in \mathcal{M}_K$. Now $m \le (y - E_n - \beta)/K$. Thus we have $y \ge Km + E_n + \beta = H_m + \beta + E_n - E_m > H_m + \beta$.

(c) $k \le \ell < k + t$. On the one hand, $m > (y - E_n - \beta)/K - 1$, i.e., $y < K(m + 1) + E_n + \beta$. Thus $\ell = y - H_m - \beta < K(m+1) + E_n - E_m - Km = E_n - E_m + K$. Note that both $y - H_m - \beta$ and $E_n - E_m + K$ are in $\mathcal{M}_K$, so $\ell = y - H_m - \beta \le E_n - E_m < k + t$. On the other hand, by (b), $\ell = y - H_m - \beta \ge E_n - E_m = k$. ■

Theorem 6 provides a recursive winning strategy for Γ_K in misère play, which is exponential. We now examine whether Γ_K has a poly-time winning strategy or not.

In Section 7 of [5], three characterizations, recursive, algebraic and arithmetic, are given for the P-positions of Generalized Wythoff in misère play, which is the case $K = s = 1$ of Γ_K. Take the recursive and algebraic characterizations for example, denote by $\{(E'_n, H'_n)\}_{n \ge 0}$ the P-positions of Generalized Wythoff with parameter $t \in \mathbb{Z}^+$, we have

(i) *Recursive characterization*

For $t = 1 : (E'_0, H'_0) = (2, 2)$, $E'_n = \text{mex}\,\{E'_i, H'_i \mid 0 \le i < n\}$, $H'_n = E'_n + n\ (n \ge 1)$.
For $t > 1 : E'_n = \text{mex}\,\{E'_i, H'_i \mid 0 \le i < n\}$, $H'_n = E'_n + tn + 1\ (n \ge 0)$.

(ii) *Algebraic characterization*

For $t = 1$: $(E'_0, H'_0) = (2, 2)$, $(E'_1, H'_1) = (0, 1)$,

$$E'_n = \lfloor n\phi \rfloor, \qquad H'_n = \lfloor n\phi^2 \rfloor (n \geq 2), \quad \text{where } \phi = (1 + \sqrt{5})/2.$$

For $t > 1 : E'_n = \lfloor n\alpha + \gamma \rfloor, H'_n = \lfloor n\beta + \delta \rfloor$ $(n \geq 0)$, where $\alpha = (2 - t + \sqrt{t^2 + 4})/2$, $\beta = \alpha + t$, $\gamma = 1/\alpha$, $\delta = \gamma + 1$.

For the arithmetic winning strategy, which involves a continued fraction and two numeration systems, p-system and q-system, we refer the reader to Section 7 of [5]. It was pointed out there that the first one strategy is exponential while the last two provide poly-time strategies for Generalized Wythoff. Now there exists a connection between Γ_K with parameters $K, s, t \in \mathbb{Z}^+$ and Generalized Wythoff but with parameter $t' = \lceil t/K \rceil$, t, K as in Γ_K.

Theorem 7. *Let $s = 1$. $E'_n = E_n/K$, $H'_n = H_n/K$, where $\{(E_n, H_n)\}_{n \geq 0}$ and $\{(E'_n, H'_n)\}_{n \geq 0}$ the P-generators of Γ_K and the P-positions of Generalized Wythoff.*

Proof. This follows by the same method as in the proof of Theorem 4. ■

Corollary 8. *In misère play, (x, y) is a P-position of Γ_K $(s = 1)$ if and only if $(\lfloor x/K \rfloor, \lfloor y/K \rfloor)$ is a P-position of Generalized Wythoff.*

Proof. Directly follows from Theorem 7. ■

Now based on this simple connection, together with the poly-time winning strategy for Generalized Wythoff, Γ_K has a poly-time winning strategy for $s = 1$. However, for $s > 1$, there is no poly-time winning strategy yet.

4. Rook type restrictions of (s, t)-Wythoff's game

In this section, let $\mathbb{Z}^{even} = \{2n \mid n \in \mathbb{Z}^0\}$, $\mathbb{Z}^{odd} = \{2n + 1 \mid n \in \mathbb{Z}^0\}$. Let

$$\delta_n = \begin{cases} 0, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases}$$

4.1. The P-positions of Γ_{OA}

In Γ_{OA}, asymmetry of the game rules implies that (a, b) is not necessarily identical to (b, a).

Theorem 9. *Let $s, t \in \mathbb{Z}^+$. For Γ_{OA} in normal play,*

(1) If $s = t = 1$, then $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(2n, 0), (2n + 1, 2)\}$.

(2) If $s + t > 2$, then $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(A_n, B_n)\}$, where $A_n = n$, $B_n = \delta_n(sn + (n + 1)t/2)$.

Proof. (1) Clearly for the stability property of $\mathscr{P}$. Suppose (a, b) is a position not in $\mathscr{P}$. If $a = 2n$ for some $n \in \mathbb{Z}^0$, move $(a, b) \to (2n, 0)$. If $a = 2n + 1$ for some $n \in \mathbb{Z}^0$, then $b \in \{0, 1\}$ or $b \geq 3$. For the former, we move $(a, b) \to (2n, 0)$. Otherwise, move $(a, b) \to (2n + 1, 2)$.

(2)

Proof of Fact I. Given $(A_n, B_n) \in \mathscr{P}$. Suppose that $(A_n, B_n) \to (A_m, B_m) \in \mathscr{P}$. Then $n \in \mathbb{Z}^{even}$ cannot happen, since $B_n = 0 < B_m$. Thus we have $n \in \mathbb{Z}^{odd}$. If m is also odd, then $k = n - m \geq 2$, thus $\ell = B_n - B_m = s(n-m) + (n-m)t/2 \geq sk + t$, which contradicts the condition $0 < k \leq \ell < sk + t$. But if m is even, then we have $k = n - m > 0$ and $\ell = B_n = sn + (n + 1)t/2 \geq sk + t$, another contradiction.

Proof of Fact II. Let (x, y) be a position not in $\mathscr{P}$. If x is even, then move $y \to 0$. If x is odd, there exists some n such that $x = A_n = n$ and we have either $y > B_n$ or $0 \leq y < B_n$. If $y > B_n$, then move $y \to B_n$. If $0 \leq y < B_n$ we distinguish the following four cases:

• $y = 0$. Then move $(x, y) \to (x - 1, 0)$.

• $1 \leq y < x$. We move $(x, y) \to (x - y - \delta_y + 1, 0) \in \mathscr{P}$ on account of $x - y - \delta_y + 1 \in \mathbb{Z}^{even}$. This move is legal, since $k = y + \delta_y - 1 > 0$, $\ell = y > 0$, and $0 \leq \ell - k \leq 1 < (s - 1)k + t$.

• $x \leq y < sx + t$. Then move $(x, y) \to (0, 0)$, which satisfies the condition Eq. (1) with $k = A_n$, $\ell = y$.

• $sx + t \leq y < B_n$. Put $m = 2\lfloor (y - sx - t)/t \rfloor + 1$ and move $(x, y) \to (A_m, B_m)$. This move is legal, since (a) $m < n$, (b) $y > B_m$, (c) $A_n - A_m \leq y - B_m < s(A_n - A_m) + t$. Indeed,

(a) $y - sx - t < B_n - sx - t = (n - 1)t/2$, so $m \leq 2(y - sx - t)/t + 1 < n$;

(b) $m \leq 2(y - sx - t)/t + 1$, so $y \geq (m - 1)t/2 + sx + t = B_m + s(n - m) > B_m$;

(c) $m > 2((y - sx - t)/t - 1) + 1 = 2(y - sx - t)/t - 1$, so $y < (m + 1)t/2 + sx + t = sn + (m + 3)t/2$; by (b), $y - B_m \geq n - m = A_n - A_m$, hence,

$$A_n - A_m \leq y - B_m < sn + (m + 3)t/2 - sm - (m + 1)t/2 = s(A_n - A_m) + t.$$

Thus Eq. (1) is satisfied. ■

Theorem 10. *Put $s, t \in \mathbb{Z}^+$. For Γ_{OA} in misère play,*
(1) *If $s = t = 1$, then $\mathscr{P} = (0, 1) \cup \bigcup_{n=0}^{\infty}\{(2n+1, 0), (2n+2, 2)\}$.*
(2) *If $s + t > 2$, then $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(E_n, H_n)\}$, where $E_n = n$, $H_n = (1 - \delta_n)(sn + tn/2 + 1)$.*

Proof. (1) Both stability and absorbing properties of $\mathscr{P}$ when $s = t = 1$ are simple. The details are left to the reader.
(2)

Proof of Fact I. Suppose a move from (E_n, H_n) produces another position of the form (E_m, H_m). It is easy to see that the only possibility is that n is even. If m is also even, this implies $k = n-m \geq 2$, then $\ell = H_n - H_m = s(n-m)+t(n-m)/2 \geq sk+t$, which contradicts Eq. (1). If n is even but m is odd, then $k = n-m > 0$, thus $\ell = H_n - H_m = sn+tn/2+1 \geq s(n-m)+tn/2 \geq sk + t$, another contradiction.

Proof of Fact II. Let (x, y) be a position not in $\mathscr{P}$. We will show that there exists a legal move such that $(x, y) \to (E_n, H_n)$. Put $x = E_n = n$ for some $n \in \mathbb{Z}^+$. If $x = 0$, then $(E_0, H_0) = (0, 1)$. For $(0, 0)$, the next player wins without doing anything; for $y > 1$, we only need to move $y \to 1$. If x is odd, then move $y \to 0 = H_n$. If x is even, this implies $y > H_n$ or $0 \leq y < H_n$. For the former, we move $y \to H_n$; while for the latter, we distinguish the following four cases:

- $y = 0$. Then move $(x, y) \to (E_n - 1, 0) \in \mathscr{P}$.
- $1 \leq y \leq x$. In this case, move $(x, y) \to (x - y - \delta_y + 1, 0) \in \mathscr{P}$, since $x - y - \delta_y + 1 > 0$ is odd. This move is legal: (a) $k = y - 1 + \delta_y > 0$, (b) $\ell = y > 0$, (c) $0 \leq \ell - k \leq 1 < (s-1)k + t$.
- $x < y < sx + t + 1$. we move $(x, y) \to (E_0, H_0) = (0, 1)$, which satisfies Eq. (1) with $k = x$, $\ell = y - 1 < sx + t$.
- $sx + t + 1 \leq y < H_n$. Put $m = 2\lfloor (y - sx - 1)/t \rfloor$ and move $(x, y) \to (E_m, H_m)$. This is a legal move, since (a) $m < n$, (b) $y > H_m$, and (c) $E_n - E_m \leq y - H_m < s(E_n - E_m) + t$. Indeed,

(a) $y - sx - 1 < H_n - sn - 1 = nt/2$, so $m = 2\lfloor (y - sx - 1)/t \rfloor \leq 2(y - sx - 1)/t < n$;
(b) $m \leq 2(y - sx - 1)/t$, so $y \geq mt/2 + sx + 1 = H_m + s(n - m) > H_m$;
(c) $m > 2(y - sx - 1)/t - 2$, thus $y < sn + (m+2)t/2 + 1$; by (b), $y - H_m \geq n - m = E_n - E_m$.
Therefore, $E_n - E_m \leq y - H_m < sn + mt/2 + t + 1 - sm - mt/2 - 1 = s(E_n - E_m) + t$, thus Eq. (1) is satisfied. ■

4.2. The P-positions of Γ_{OO}

Obviously, the game rules of Γ_{OO} are symmetrical, so we say (a, b) is a P-position, meaning that (b, a) is also a P-position.

Theorem 11. *Given $s, t \in \mathbb{Z}^+$. For Γ_{OO} in normal play, $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(0, 2n)\}$.*

Proof. A move from $(0, 2n)$ clearly leads to a position not in $\mathscr{P}$. Let (x, y) with $x \leq y$ be a position not in $\mathscr{P}$. If $x = 0$ and y is odd, only move $y \to y - 1$. Consider $x > 0$. If $x, y \in \mathbb{Z}^{odd}$ or $x, y \in \mathbb{Z}^{even}$, then move $(x, y) \to (0, y - x) \in \mathscr{P}$. Otherwise, we take the entire pile with an odd number of tokens. ■

Theorem 12. *Given $s, t \in \mathbb{Z}^+$. For Γ_{OO} in misère play,*

$$\mathscr{P} = \begin{cases} \{(0, 2n+1), (2, 2n) \mid n \in \mathbb{Z}^+\}, & \text{if } s = t = 1, \\ \{(0, 2n+1) \mid n \in \mathbb{Z}^+\}, & \text{if } s + t > 2. \end{cases}$$

Proof. The stability property of $\mathscr{P}$ is straightforward. Let (x, y) with $x \leq y$ be a position not in $\mathscr{P}$. It suffices to show that from (x, y) there is a move terminating in $\mathscr{P}$. Consider three cases:

- $x = 0$. Clearly for $y = 0$. If $y > 0$, then y is even and move $y \to y - 1$.
- $x = 1$. Then move $(1, y) \to (0, y - 1 + \delta_y) \in \mathscr{P}$, since $y - 1 + \delta_y$ is odd.
- $x \geq 2$. For $s = t = 1$. If $x = 2$, then move $(x, y) \to (2, y-1)$; if $x \geq 3$, we move $(x, y) \to (2-2\delta_{y-x},\ y-x-2\delta_{y-x}+2)$ by taking $x+2\delta_{y-x}-2 > 0$ tokens from both piles. Note that if $y-x$ is odd, we have $(2-2\delta_{y-x},\ y-x-2\delta_{y-x}+2) = (0, y-x) \in \mathscr{P}$; if $y - x$ is even, then $(2 - 2\delta_{y-x},\ y - x - 2\delta_{y-x} + 2) = (2, y - x + 2) \in \mathscr{P}$.

For $s + t > 2$, we move $(x, y) \to (0, y - x - \delta_{y-x} + 1) \in \mathscr{P}$, since $y - x - \delta_{y-x} + 1$ is odd. This is a legal move, since: (a) $k = x - 1 + \delta_{y-x} > 0$, (b) $\ell = x$, and (c) $0 \leq \ell - k = 1 - \delta_{y-x} \leq 1 < s + t - 1 \leq (s-1)k + t$. ■

4.3. The P-positions of Γ_{OE}

In Γ_{OE}, (a, b) is not necessarily identical to (b, a) because of asymmetry.

Theorem 13. *Let $s = t = 1$. For Γ_{OE} in normal play,*

$$\mathscr{P} = \bigcup_{n=0}^{\infty}\{(2n, 0), (2n, 1), (2n+1, 4n+3), (2n+1, 4n+4)\}.$$

Proof. The proof of the stability property of $\mathscr{P}$ is simple, we leave the details to the reader. Now we prove the absorbing property of $\mathscr{P}$. Let (x, y) be a position not in $\mathscr{P}$.

If x is even, then move $(x, y) \to (x, \delta_y)$.

Table 5
The first few P-positions of Γ_{OE} for $s = 2, t = 3$ in normal play.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	1	7	13	19	25	31	37	43	49	55	61	67	73	79	85
B'_n	0	6	0	18	0	30	0	42	0	50	0	66	0	78	0

If $x = 2n + 1$ for $n \in \mathbb{Z}^0$. Then $y \geq 4n + 5$ or $0 \leq y \leq 4n + 2$. For the former, we move $(x, y) \to (2n + 1, 4n + 4 - \delta_y)$. For the latter, if $y = 0$, then move $(x, y) \to (2n, 0)$; if $1 \leq y \leq x + 1$, then move $(x, y) \to (x - y - \delta_y + 1, 1 - \delta_y)$ by taking $y + \delta_y - 1 > 0$ tokens from both piles. Since $x - y - \delta_y + 1$ is even and $1 - \delta_y \in \{0, 1\}$, thus $(x - y - \delta_y + 1, 1 - \delta_y) \in \mathscr{P}$. Finally, if $x + 2 \leq y \leq 4n + 2$. Then we move $(x, y) \to (y + \delta_y - x - 2, 2y + \delta_y - 2x - 2)$ by taking $2x - y - \delta_y + 2$ $(\geq 2 - \delta_y > 0)$ tokens from both piles. The proof is completed by showing that $(y + \delta_y - x - 2, 2y + \delta_y - 2x - 2) \in \mathscr{P}$:

Let $y + \delta_y - x - 2 = \phi$. Then $2y + \delta_y - 2x - 2 = 2\phi + 2 - \delta_y \in \{2\phi + 1, 2\phi + 2\}$. Since $y + \delta_y$ is even, x is odd, we get ϕ is odd. It is easy to see that $(\phi, 2\phi + 1), (\phi, 2\phi + 2) \in \mathscr{P}$. ■

Theorem 14. *Let $s + t > 2$. For Γ_{OE} in normal play, $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(A_n, B_n), (A_n, B'_n)\}$, where for $n \geq 0$,*

$$\begin{cases} A_n = n, \\ B_n = (s + t + 1)A_n + 1, \\ B'_n = \delta_n(B_n - 1). \end{cases}$$

Example 5. For $s = 2, t = 3$, we display the first few P-positions of Γ_{OE} in Table 5.

Proof. Proof of Fact I. Given $(A_n, B_n) \in \mathscr{P}$. Suppose that $(A_n, B_n) \to (A_m, B_m) \in \mathscr{P}$. Then we have $k = n - m > 0$, and $\ell = (s + t + 1)(n - m) > sk + t$, which contradicts Eq. (1).

Suppose that $(A_n, B_n) \to (A_m, B'_m) \in \mathscr{P}$. In this case, we have $k = n - m > 0$, and

$$\begin{aligned} \ell = B_n - B'_m &= \begin{cases} (s + t + 1)(n - m) + 1 & \text{if } m \in \mathbb{Z}^{odd} \\ (s + t + 1)n + 1 & \text{if } m \in \mathbb{Z}^{even} \end{cases} \\ &> (s + t + 1)k > sk + t, \end{aligned}$$

also contradicting Eq. (1).

Given $(A_n, B'_n) \in \mathscr{P}$. Notice that if n is even and so $B'_n = 0$, then any move from $(A_n, 0)$ cannot lead to a position in $\mathscr{P}$. Now suppose n is odd and so $B'_n = B_n - 1$. If $(A_n, B'_n) \to (A_m, B_m) \in \mathscr{P}$, then we have $k = n - m > 0$, and $\ell = B_n - B_m - 1 = (s + t + 1)(n - m) - 1 = (s + t + 1)k - 1 \geq sk + t$, a contradiction; if $(A_n, B'_n) \to (A_m, B'_m) \in \mathscr{P}$, then we get $k = n - m > 0$, and

$$\begin{aligned} \ell = B_n - 1 - B'_m &= \begin{cases} (s + t + 1)(n - m) & \text{if } m \in \mathbb{Z}^{odd} \\ (s + t + 1)n & \text{if } m \in \mathbb{Z}^{even} \end{cases} \\ &\geq (s + t + 1)k > sk + t, \end{aligned}$$

another contradiction.

Proof of Fact II. Let (x, y) be a position not in $\mathscr{P}$. We show that there exists a legal move such that $(x, y) \to (A_n, B_n)$ or (A_n, B'_n).

Put $x = A_n$ for some $n \in \mathbb{Z}^0$. We distinguish two cases: (i) x is even; (ii) x is odd.

Case (i) $x = A_n = n$ is even.

In this case, note first that $B_n = (s + t + 1)n + 1$ is odd and $B'_n = 0$. The fact $(x, y) \notin \mathscr{P}$ implies that $y > B_n$ or $0 < y < B_n$. For $y > B_n$, if y is even, then move $y \to B'_n$; if y is odd, we move $y \to B_n$. For $0 < y < B_n$, we proceed by distinguishing three subcases:

• $1 \leq y < x$. Then move $(x, y) \to (x - y - \delta_y, 0) \in \mathscr{P}$. This move is legal, since (a) $k = y \geq 1$, (b) $\ell = y + \delta_y \geq 2$, (c) $0 \leq \ell - k = \delta_y \leq 1 < (s - 1)k + t$.

• $x \leq y \leq B_n - 2$. In this subcase, put $m = \lfloor (y - x)/(s + t) \rfloor$ and move $(x, y) \to (A_m, B_m)$. This move is legal:
(a) $0 \leq m < n$. Indeed, $0 \leq y - x \leq B_n - x - 2 = (s + t)n - 1 < (s + t)n$, so $0 \leq m = \lfloor (y - x)/(s + t) \rfloor \leq (y - x)/(s + t) < n$.
(b) By the definition of m, we have $(y - x)/(s + t) - 1 < m \leq (y - x)/(s + t)$, i.e.,

$$(s + t)m \leq y - x < (s + t)(m + 1). \tag{7}$$

Thus, $y \geq (s + t)m + x = B_m + (A_n - A_m) - 1 \geq B_m$ by virtue of $A_n - A_m \geq 1$.

If $y = B_m$ then $A_n - A_m = 1$. This is a legal move only from the first pile.

If $y - B_m \geq 1$, then it follows from Eq. (7) that $|(y - B_m) - (x - A_m)| = |y - x - (s + t)m - 1| < s + t - 1 \leq (s - 1)\lambda + t$, where $\lambda := \{A_n - A_m, y - B_m\} \geq 1$.

• $y = B_n - 1$. Then move $y \to 0$ by taking $y \in \mathbb{Z}^{even}$ tokens from the second pile.

Table 6
The first few P-positions of Γ_{OE} for $s = t = 3$ in misère play.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
H_n	2	0	9	15	20	25	30	35	40	45	50	55	60	65	70
H'_n	3	1	10	0	19	0	29	0	39	0	49	0	59	0	64

Case (ii) $x = A_n = n$ is odd.

In this case, $B'_n = B_n - 1$, then $y > B_n$ or $0 \le y \le B_n - 2$. For $y > B_n$. we see $B_n = (s + t + 1)n + 1$ is odd if $s + t$ is odd, or B_n is even if $s + t$ is even. Thus if $(y, s + t \in \mathbb{Z}^{odd})$ or $(y, s + t \in \mathbb{Z}^{even})$, then we move $y \to B_n$; if $(y \in \mathbb{Z}^{even}, s + t \in \mathbb{Z}^{odd})$ or $(y \in \mathbb{Z}^{odd}, s + t \in \mathbb{Z}^{even})$, we move $y \to B'_n$. For $0 \le y \le B_n - 2$. We consider the following three subcases:

• $y = 0$. We just move $(x, y) \to (A_n - 1, 0) \in \mathscr{P}$.

• $1 \le y < x$. In this subcase, we move $(x, y) \to (x - y - 1 + \delta_y, 0) \in \mathscr{P}$. This is a legal move, since (a) $k = y > 0$, (b) $\ell = y + 1 - \delta_y > 0$, (c) $0 \le \ell - k = \delta_y \le 1 < (s - 1)k + t$.

• $x \le y \le B_n - 2$. We move $(x, y) \to (A_m, B_m)$ with $m = \lfloor (y - x)/(s + t) \rfloor$. This follows from the same method as in case (i). ∎

Theorem 15. *Let $s = t = 1$. For Γ_{OE} in misère play,*

$$\mathscr{P} = \{(0, 2), (0, 3), (2, 3), (2, 6)\} \cup \bigcup_{n=0}^{\infty} \left\{ \begin{matrix} (2n + 1, 0), (2n + 1, 1), \\ (2n + 4, 4n + 9), (2n + 4, 4n + 10) \end{matrix} \right\}.$$

Proof. The stability property of $\mathscr{P}$ is simple. We are left with the task of proving the absorbing property of $\mathscr{P}$. It is easy to check that $(0, 2), (0, 3), (2, 3), (2, 6)$ are all P-positions by the knowledge of Example 1 in Section 2. If $x = 2n + 1$ for some $n \in \mathbb{Z}^0$, then move $(x, y) \to (x, \delta_y)$. If $x = 2n + 4$ for some $n \in \mathbb{Z}^0$, then $y > 4n + 10$ or $0 \le y < 4n + 9$. For the former, we move $(x, y) \to (2n + 4, 4n + 9)$ (if y is odd) or $(x, y) \to (2n + 4, 4n + 10)$ (if y is even). For the latter, we consider three cases:

• $0 \le y \le x$. If $y = 0$, then move $x \to 2n + 1$, or else, we move $(x, y) \to (x - y - \delta_y + 1, 0) \in \mathscr{P}$, which satisfies Eq. (1) with $k = y + \delta_y - 1$ and $\ell = y$.

• $x < y \le x+4$. If $y \in \{x+1, x+4\}$, then remove $x-2$ tokens from both piles leading to $(2, 3)$ or $(2, 6)$; if $y \in \{x+2, x+3\}$, remove x tokens from both piles leading to $(0, 2)$ or $(0, 3)$.

• $x + 5 \le y < 4n + 9$. Then move $(x, y) \to (y - x - 2 + \delta_y, 2y - 2x - 2 + \delta_y)$ by taking $2x - y + 2 - \delta_y$ tokens from both piles. Let $y - x - 2 + \delta_y = \phi$. Clearly ϕ is even and because of $\phi \ge 3 + \delta_y$, then $\phi \ge 4$. Thus there exists some $n \in \mathbb{Z}^0$ such that $\phi = 2n + 4$. Furthermore, $2y - 2x - 2 + \delta_y = 2\phi + 2 - \delta_y \in \{2\phi + 1, 2\phi + 2\} = \{4n + 9, 4n + 10\}$. Hence, $(y - x - 2 + \delta_y, 2y - 2x - 2 + \delta_y) \in \mathscr{P}$. ∎

Theorem 16. *Let $s + t > 2$. For Γ_{OE} in misère play, $\mathscr{P} = \bigcup_{n=0}^{\infty}\{(E_n, H_n), (E_n, H'_n)\}$, where for $n \in \{0, 1, 2\}$,*

n	0	1	2
E_n	0	1	2
H_n	2	0	$2s+t+3$
H'_n	3	1	$2s+t+4$

and for $n \ge 3$,

$$\begin{cases} E_n = n, \\ H_n = (s + t + 1)E_n + 2 - \delta_s - t, \\ H'_n = (1 - \delta_n)(H_n - 1). \end{cases} \tag{8}$$

Example 6. For $s = t = 3$, we display the first few P-positions of Γ_{OE} in Table 6.

Proof. The proof is tedious. We first prove the stability property of $\mathscr{P}$. Given a position (E_n, H_n) (or (E_n, H'_n)) in $\mathscr{P}$, if $n < 3$, we leave it to the reader to verify that a legal move from (E_n, H_n) (or (E_n, H'_n)) leads to a position not in $\mathscr{P}$. For $n \ge 3$, it is easy to check that a legal move from (E_n, H_n) (or (E_n, H'_n)) cannot land in $\bigcup_{i<3}\{(E_i, H_i), (E_i, H'_i)\}$. Let $m \ge 3$.

Now suppose $(E_n, H_n) \to (E_m, H_m)$, then $k = n - m > 0$ and $\ell = H_n - H_m = (s + t + 1)(n - m) > sk + t$, which contradicts Eq. (1).

Suppose that $(E_n, H_n) \to (E_m, H'_m)$, then $k = n - m > 0$. And if m is odd, $\ell = H_n - 0 = (s + t + 1)n + 2 - \delta_s - t > (n - m)s + (n - 1)t > sk + t$; if m is even, then $\ell = H_n - (H_m - 1) = (s + t + 1)(n - m) + 1 > sk + t$. Both contradict Eq. (1).

Next suppose $(E_n, H'_n) \to (E_m, H_m)$. It is impossible that n is odd. Indeed, if so, it follows by the definition of δ_n that $H'_n = 0 < H_m$. If n is even, then we have $H'_n = H_n - 1$ and $k = n - m > 0$, but $\ell = (H_n - 1) - H_m = (s+t+1)(n-m) - 1 \geq sk+t$, another contradiction.

Finally, suppose that $(E_n, H'_n) \to (E_m, H'_m)$. As above, n is even. If m is also even, then $\ell = (H_n - 1) - (H_m - 1) = (s+t+1)(n-m) > sk+t$; if n is even but m is odd, then $\ell = H_n - 1 = (s+t+1)n + 1 - \delta_s - t > (n-m)s + (n-1)t > sk+t$. In a word, this move is also illegal.

We next prove the absorbing property of $\mathscr{P}$. Let (x, y) be a position not in $\mathscr{P}$. Put $x = E_n = n$ for some $n \in \mathbb{Z}^0$.

If $x = 0$, then $y \in \{0, 1\}$ or $y \geq 4$. Obviously, $(0, 0)$ and $(0, 1)$ are N-positions. For $y \geq 4$, then move $(0, y) \to (0, 2+\delta_y) \in \mathscr{P}$.

If $x = 1$, then $y \geq 2$, move $(1, y) \to (1, \delta_y) \in \mathscr{P}$.

If $x = 2$, we have either $y > 2s + t + 4$ or $0 \leq y < 2s + t + 3$.

Case (i) $y > 2s + t + 4$. If ($y \in \mathbb{Z}^{odd}$ and $t \in \mathbb{Z}^{even}$) or ($y \in \mathbb{Z}^{even}$ and $t \in \mathbb{Z}^{odd}$), we move $(2, y) \to (2, 2s + t + 3) \in \mathscr{P}$ since $\ell = y - 2s - t - 3 > 0$ is even; if ($y, t \in \mathbb{Z}^{odd}$) or ($y, t \in \mathbb{Z}^{even}$), then we move $(2, y) \to (2, 2s + t + 4) \in \mathscr{P}$ because $y - 2s - t - 4$ is always even.

Case (ii) $0 \leq y < 2s + t + 3$. If $y = 0$, we move $(2, 0) \to (1, 0)$; if $y \in \{1, 2, 3\}$, we move $(2, y) \to (1, 1)$; if $y = 4$, move $(2, 4) \to (0, 2)$, if $5 \leq y < 2s + t + 3$, then move $(2, y) \to (0, 3)$, which satisfies Eq. (1) with $k = 2$ and $\ell = y - 3$.

If $x \geq 3$, we proceed by distinguish two cases:

Case (iii) $x = n$ is odd.

In this case, $H'_n = 0$ and so $y > H_n$ or $0 < y < H_n$. For $y > H_n$, if y is even, we move $y \to 0$; if y is odd, then move $y \to H_n$ as $\ell = y - H_n = y - (n-1)(s+t) - (s - \delta_s) - n + 2$ is even. The case $0 < y < H_n$ is rebarbative. With patience we proceed by distinguishing seven subcases:

• $0 < y < x$. If y is even, we move $y \to H'_n = 0$; if y is odd, we move $(x, y) \to (x - y - 1, 0)$ since $x - y - 1$ is odd. Obviously this move satisfies Eq. (1) with $k = y$ and $\ell = y + 1$.

• $y \in \{x, x + 1\}$. Then move $(x, y) \to (1, 1)$.

• $x + 2 \leq y \leq x + 2s + t$. We move $(x, y) \to (0, 3)$, which is legal, since (a) $k = x > 0$, (b) $\ell = y - 3 \geq x - 1 > 0$, (c) $|\ell - k| \leq 2s + t - 3 < 2(s - 1) + t \leq (s - 1)\lambda + t$, where $\lambda := \min\{x, y - 3\} \geq 2$.

• $y = x + 2s + t + 1$. We move $(x, y) \to (2, 2s + t + 3) \in \mathscr{P}$ by removing $x - 2 > 0$ tokens from both piles.

• $x + 2s + t + 2 \leq y \leq x + 3s + 2t$. Then move $(x, y) \to (2, 2s + t + 4) \in \mathscr{P}$. This move is legal, since (a) $k = x - 2 > 0$, (b) $\ell = y - (2s + t + 4) \geq x - 2 > 0$, (c) $|\ell - k| \leq s + t - 2 < s + t - 1 \leq (s - 1)k + t$.

• $x + 3s + 2t + 1 \leq y < H_n - 1$. Put $m = \lfloor (y - x + t - 1 + \delta_s)/(s + t) \rfloor$ and move $(x, y) \to (E_m, H_m)$. This move is also legal, since

(a) $n > m \geq 3$. Indeed, $y - x + t - 1 + \delta_s < H_n - x + t - 2 + \delta_s = (s+t)n$, thus we have $m \leq (y - x + t - 1 + \delta_s)/(s + t) < n$. On the other hand, $y - x + t - 1 + \delta_s \geq x + 3s + 2t + 1 - (x - t + 1 - \delta_s) \geq 3(s + t)$. so $m \geq 3$ and $k = n - m > 0$.

(b) $y \geq H_m$. By the definition of m, $(y - x - s - 1 + \delta_s)/(s + t) < m \leq (y - x + t - 1 + \delta_s)/(s + t)$, i.e.,

$$(s + t)m - t + 1 - \delta_s \leq y - x < (s + t)m + s + 1 - \delta_s. \tag{9}$$

Thus $y \geq (s + t)m + x - t + 1 - \delta_s = H_m + (E_n - E_m) - 1 \geq H_m$ by virtue of $E_n - E_m \geq 1$.

If $y = H_m$, then $E_n - E_m = 1$. This is a legal move only from the first pile.

If $y - H_m \geq 1$, then it follows from Eq. (9) that $|(y - H_m) - (x - E_m)| = |y - x - (s+t)m - 2 + t + \delta_s| < s + t - 1 \leq (s - 1)\lambda + t$, where $\lambda := \min\{E_n - E_m, y - H_m\} \geq 1$.

• $y = H_n - 1$. Note that $H_n - 1 = (n - 1)(s + t) + (s - \delta_s) + n + 1$ is even on account of $n \in \mathbb{Z}^{odd}$ and $s - \delta_s \in \mathbb{Z}^{even}$. Thus we move simply $y \to H'_n = 0$.

Case (iv) $x = n$ is even.

In this case, $H'_n = H_n - 1$ and so we have either $y > H_n$ or $0 \leq y \leq H_n - 2$.

For $y > H_n$. It is worth to note that if $s + t$ is odd, then $t + \delta_s$ is also odd, thereby $H_n = (s + t + 1)n + 2 - t - \delta_s$ is odd; if $s + t$ is even, meaning that $t + \delta_s$ is also even, and so H_n is even. Therefore, if ($y, s + t \in \mathbb{Z}^{odd}$) or ($y, s + t \in \mathbb{Z}^{even}$), then we move $y \to H_n$ since $y - H_n$ is even; if ($y \in \mathbb{Z}^{odd}$ and $s + t \in \mathbb{Z}^{even}$) or ($y \in \mathbb{Z}^{even}$ and $s + t \in \mathbb{Z}^{odd}$), then we move $y \to H'_n$ because $y - H'_n = y - H_n + 1$ is still even.

For $0 \leq y \leq H_n - 2$. If $y = 0$, we move $(x, 0) \to (x - 1, 0)$. If $1 \leq y \leq x - 1$, we move $(x, y) \to (x - y - 1 + \delta_y, 0) \in \mathscr{P}$ with $x - y - 1 + \delta_y$ being odd, which satisfies Eq. (1) with $k = y$ and $\ell = y + 1 - \delta_y$. Otherwise, analysis for $x \leq y \leq H_n - 2$ is the same as the proof of case (iii), more details are left to the reader. ∎

Remark 1. Similar to Γ_{OE}, maybe we can define Γ_{EO}, *Even–Odd-Nim* (s, t)*-Wythoff's game*: A player chooses the first pile and takes *even* $k > 0$ tokens, or chooses the second pile and takes *odd* $\ell > 0$ tokens, the move rules are the same with (s, t)-Wythoff's Game when moving from both piles. The move rules of these two games imply that (x, y) is a P-position of Γ_{EO} if and only if (y, x) is a P-position of Γ_{OE}. Thus the P-positions of Γ_{EO} are easily obtained by Theorems 13–16 with (x, y) replaced by (y, x).

5. Conclusion

In this paper, the game Γ_K is defined and completely solved for any $K, s, t \in \mathbb{Z}^+$ in both normal and misère play. It is a generalization of both the original (s, t)-Wythoff's game and EEW investigated in [12]. Both exponential and polynomial winning strategies for Γ_K are given in both normal and misère play. However, in misère play, whether Γ_K has a polynomial time winning strategy or not is still open for all $s > 1$.

Following this, Γ_{OA}, Γ_{OO}, and Γ_{OE} are investigated. Under both normal and misère play conventions, the sets of P-positions of these three games are given algebraically for all $s, t \geq 1$. Motivated by these games, we may associate additional interesting games, for instance:

Open problem. Define Γ_{EE} (Even–Even-Nim (s, t)-Wythoff's game): a player may only remove an *even* (> 0) number of tokens when moving from a single pile, and the move rules remain unchanged when moving from both piles. This game is also a rook type restriction of (s, t)-Wythoff's game. Determine the P-positions of Γ_{EE}.

Further, what are the P-positions if a player can remove a multiple of K $(\in \mathbb{Z}^+)$ tokens when moving from one pile (a generalization of Γ_{EE})? And what if a player is restricted to take k tokens, with $k \in \{nK + 1 : n \in \mathbb{Z}^+\}$ (or $k \in \{nK + K - 1 : n \in \mathbb{Z}^+\}$), when moving from one pile (a generalization of Γ_{OO}, which is precisely the case $K = 2$)?

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