On Invariant Games

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We assume that the reader has some knowledge about combinatorial game theory. Basic definitions can be found in [1], [2].

A *take-aqway* game is played with finitely many tokens distributed into k piles. Such a game has been called invariant in [4] if the same moves can be played from any game position, provided only that there are enough tokens in the piles. In the sequel any take-away game may be called simply *game*.

It is convenient to code positions and moves of games by k-tuples of nonnegative integers. For two k-tuples $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$, we write $x \prec y$ if $x_i \le y_i$ for all $i = 1, \ldots, k$.

Definition 1. A move $x = (x_1, \ldots, x_k)$ in a game G is *invariant*, if it can be played from every position $p = (p_1, \ldots, p_k)$ for which $x \prec p$. Otherwise the move x is *variant.*

So a variant move can be played from some game positions but there exists a game position from which it cannot be played.

Definition 2. [4] A game G is *invariant* if all its moves are invariant. It is *variant* if it has some variant move.

Remark 1. A game can be described by a digraph $G = (V, E)$, where V is the set of game positions, and E its set of moves. The P-positions constitute the unique kernel K of G: It is an independent set (no edges), and absorbing (every vertex $\notin K$ has an edge leading into K). It follows that the only variant moves x are those for which there exist vertices $p, q \in K$ such that x connects p and q. We denote the set of all P-positions of a game by $\mathcal{P}$, the set of all its N-positions by $\mathcal{N}$.

A general pair of *complementary Beatty sequences* has the form $(A_n, B_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)_{n\ge1}$, where α, β are positive irrationals satisfying $\alpha^{-1} + \beta^{-1}$. Without loss of generality we may assume $\alpha < \beta$. Then actually $1 < \alpha < 2 < \beta$.

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Let $\zeta > 0$ be irrational, $m > n > 0$ integers. Then

$$(m-n)\zeta - 1 < \lfloor m\zeta \rfloor - \lfloor n\zeta \rfloor < (m-n)\zeta + 1.$$

Thus,

$$\lfloor (m-n)\zeta \rfloor \le \lfloor m\zeta \rfloor - \lfloor n\zeta \rfloor \le \lfloor (m-n)\zeta \rfloor + 1. \tag{1}$$

Duchêne and Rigo [4] asked the following question, motivated by [3]: given any sequence $S : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}^k$, is there an invariant game having S as its set of P-positions? They answered this in the negative: consider any sequence $S = (A_n, B_n)_{n \ge 0}$ beginning with $(0,0), (1,2), (3,5), (4,6)$, such that $\{A_n : n \ge 1\}$ and $\{B_n : n \ge 1\}$ split $\mathbb{Z}_{\ge 1}$. Then from the N-position $(1,1)$ one must move to $(0,0)$. Hence $(1,1)$ is a move. But playing from $(4,6)$ to $(3,5)$ is not permitted, since the P-positions constitute an independent set. They put forward the following intriguing

Conjecture 1. *Given any pair of complementary Beatty sequences $S = (A_n, B_n)_{n \ge 1}$, there exists an invariant game having $S \cup \{(0,0)\}$ as its set of P-positions.*

They then defined an invariant game whose P-positions are complementary Beatty sequences, in support of their conjecture.

It is our purpose to prove this conjecture. We state this formally:

Theorem 1. *Given any pair of complementary Beatty sequences $S = (A_n, B_n)_{n \ge 1}$, there exists an invariant game having $S \cup \{(0,0)\}$ as its set of P-positions.*

Proof. Suppose that the assertion is false. By Remark 1, there exists a position $(a,b) \in \mathcal{N}$ such that for every move x from (a,b) into $\mathcal{P}$, x also connects two P-positions. Since $1 < \alpha < 2 < \beta$, the sequences A_n, B_n are increasing. Therefore any move from a *single* pile is invariant. Thus $a > 0$, $b > 0$, and every variant move must decrease both piles. Hence for all $(i,j) \in \mathbb{Z}_{\ge 1}^2$ with $i \le a$, $j \le b$ and every $n \in \mathbb{Z}_{\ge 0}$ such that $(a-i, b-j) = (A_n, B_n)$ or $(a-i, b-j) = (B_n, A_n)$, there exist $\ell > k \ge 0$ such that $(A_\ell - i, B_\ell - j) = (A_k, B_k)$, or $(A_\ell - i, B_\ell - j) = (B_k, A_k)$. In particular, either
I(i) $(A_\ell - a, B_\ell - b) = (A_k, B_k)$, $a \le b$ or
I(ii) $(A_\ell - a, B_\ell - b) = (B_k, A_k)$, $a \le b$ or
II(i) $(A_\ell - a, B_\ell - b) = (A_k, B_k)$, $a > b$ or
II(ii) $(A_\ell - a, B_\ell - b) = (B_k, A_k)$, $a > b$.

I Since the pair of Beatty sequences partitions $\mathbb{Z}_{\ge 1}$, we have either $a = B_m$ or $a = A_m$ for some $m \in \mathbb{Z}_{\ge 1}$. If $a = B_m$, then $b \ge B_m \ge A_m$, since $\beta > \alpha$. But $b = A_m$ is impossible, since $(a,b) \in \mathcal{N}$. Thus $b > A_m$, so there is the invariant move $b \to A_m$, therefore $a = A_m$. Also, $m \le \ell$.

I(i) We have $A_\ell - A_m = A_k$. By (1), $A_{\ell-m} \le A_k \le A_{\ell-m} + 1$. Since $1 < \alpha < 2$, we might have $A_{\ell-m} + 1 = A_{\ell-m+1}$. Hence $m \in \{\ell - k, \ell - k + 1\}$.

We also have $B_\ell - b = B_k$. Suppose first that $b = B_s$ for some $s \in \mathbb{Z}_{\geq 1}$. Then $B_\ell - B_s = B_k$. By (1), $B_{\ell-s} \leq B_k \leq B_{\ell-s}+1$. Since $\beta > 2$, $B_{\ell-s}+1 = A_t$ for suitable $t \in \mathbb{Z}_{\geq 1}$. It follows that $s = \ell - k$.

If $m = \ell - k$, then $s = m$, so $b = B_m$. But then $(a, b) = (A_m, B_m) \in \mathcal{P}$, a contradiction. Hence $m = \ell - k + 1 = s + 1$, $s = m - 1$, $a = A_m$, $b = B_{m-1}$. But then can use the invariant move $A_m \to A_{m-1}$.

Therefore we must have $b = A_s$. Then $A_s = B_\ell - B_k$. By (1), $B_{\ell-k} \leq A_s \leq B_{\ell-k}+1$. By disjointness of the Beatty sequences we then have $A_s = B_{\ell-k}+1$. If $m = \ell - k$, then $b = A_s = B_m + 1$, so there is the invariant move $b \to B_m$.

So assume $m = \ell - k + 1$. Then $b = B_{m-1} + 1$, so $(a, b) = (A_m, B_{m-1} + 1)$. If $A_m - A_{m-1} = 1$, can move to (A_{m-1}, B_{m-1}) with the invariant move $(1, 1)$. So let $A_m - A_{m-1} = 2$. Consider two cases:

(A) $(A_1, B_1) = (1, t)$ $t \geq 3$. Then the first two terms of the sequence $\{A_n\}$ are 1 and 2. Therefore $\alpha < 3/2$, so $\beta > 3$. Hence $B_{n+1} - B_n \geq 3$ for all $n \in \mathbb{Z}_{\geq 1}$. This implies that $(1, 2)$ is an invariant move. So we can move $(a, b) \to (A_{m-1}, B_{m-1})$.

(B) $(A_1, B_1) = (1, 2)$. Then the moves $(a, b) \to (1, 2)$ and $(a, b) \to (2, 1)$ are moves from $(a, b) \in \mathcal{N}$ to $(1, 2) \in \mathcal{P}$ and $(2, 1) \in \mathcal{P}$ respectively. Since we assume that the game is variant, these moves also lead from some P-position to another P-position.

(B)(a) Consider the move $(a, b) \to (1, 2)$. Then for our case I(i), $(A_\ell - (a - 1), B_\ell - (b - 2)) = (A_\ell - A_m + 1, B_\ell - B_{m-1} + 1) = (A_k, B_k)$. In particular, $B_k - 1 = B_\ell - B_{m-1}$. By (1), $B_{\ell-m+1} \leq B_k - 1 \leq B_{\ell-m+1} + 1$. Equivalently, $B_k \leq B_k - 1 \leq B_k + 1$, the left hand of which is a contradiction.

(B)(b) Consider the move $(a, b) \to (2, 1)$. Then $(A_\ell - (a-2), B_\ell - (b-1)) = (A_\ell - A_m + 2, B_\ell - B_{m-1}) = (A_k, B_k)$. In particular, $A_k - 2 = A_\ell - A_m$. By (1), $A_{\ell-m} \leq A_k - 2 \leq A_{\ell-m} + 1$. Equivalently, $A_{k-1} \leq A_{k-1} - 2 \leq A_{k-1} + 1$, the left hand of which is a contradiction.

I(ii) We have $A_m = A_\ell - B_k$, $b = B_\ell - A_k$. Subtracting,

$$b - A_m = (B_\ell - A_\ell) + (B_k - A_k) \geq B_\ell - A_\ell.$$

Since $\beta > \alpha$, the sequence $\{B_n - A_n\}$ is a nondecreasing function of n. Since $m \leq \ell$ we have $B_\ell - A_\ell \geq B_m - A_m$. Thus $b \geq B_m$. But there cannot be equality, since $(a, b) \in \mathcal{N}$. Therefore $b > B_m$, so there is the invariant move $b \to B_m$.

II Analogously to case I, we deduce that $b = A_m$ for suitable $m \in \mathbb{Z}_{\geq 1}$.

II(i) We have $a = A_\ell - A_k$, $A_m = B_\ell - B_k$. Subtracting, $a - A_m = (B_k - A_k) - (B_\ell - A_\ell) \leq 0$. Thus $a \leq A_m$, a contradiction.

II(ii) We have $a = A_\ell - B_k$, $A_m = B_\ell - A_k$. Subtracting, $a - A_m = -(B_k - A_k) - (B_\ell - A_\ell) < 0$, a contradiction. ■

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