On Invariant Games

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Abstract

A 2-pile take-away game is played by two players who alternate in removing tokens from two piles according to given rules. The game is *invariant* if if the same moves can be played from any game position, provided only that there are enough tokens in the piles. Duchêne and Rigo in a paper in this issue conjectured that if the second player winning positions are given by a pair of complementary homogeneous Beatty sequences, then the game is invariant. We prove this conjecture.

Keywords: Combinatorial Game Theory, P-positions, Invariant Games

We assume that the reader has some knowledge about combinatorial game theory. Basic definitions can be found in [1], [2].

A *take-away* game is played with finitely many tokens distributed into k piles. It is played by two players, who alternate in removing tokens according to the game's rules. Throughout we consider *normal* play, that is, the player first unable to move loses (because all the piles are empty); the opponent wins. Such a game has been called invariant in [4] if the same moves can be played from any game position, provided only that there are enough tokens in the piles. In the sequel any take-away game may be called simply *game*.

It is convenient to code positions and moves of games by k-tuples of nonnegative integers. For two k-tuples $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$, we write $x \prec y$ or $y \succ x$ if $x_i \le y_i$ for all $i = 1, \ldots, k$. A *move* $x = (x_1, \ldots, x_k)$ in a game from a position $p = (p_1, \ldots, p_k)$ into a position $q = (q_1, \ldots, q_k)$ is the operation $p \to q$, done by the subtraction $p - x = (p_1 - x_1, \ldots, p_k - x_k) = q$. We also say that q is a *follower* or *option* of p, denoted by $q = F(p)$.

Definition 1. [4] A game G is *invariant* if for all positions $p = (p_1, \ldots, p_k)$ and $q = (q_1, \ldots, q_k)$ and every move $x = (x_1, \ldots, x_k)$ such that $x \prec p$, $x \prec q$, $p - x \succ 0$ and $q - x \succ 0$, the move $p \to p - x$ is permitted if and only if the move $q \to q - x$ is permitted.

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The word "permitted" in the definition doesn't mean that some arbitrary game rule doesn't permit the move, such as "never advance a pawn by three squares" in chess. The intuition is as follows. A game can be described by a digraph $G = (V, E)$, where V is the set of game positions, and E its set of moves. So $(x, y) \in E$ if and only if there is a move from position x to position y in the game. The subset $\mathcal{P} \subset V$ of P-positions constitutes the unique kernel K of G: it is an independent set (no edge between any pair of its vertices), and absorbing (every vertex $\notin K$ has an edge leading into K). The complementary subset $V \setminus \mathcal{P}$ is the set of N-positions, denoted by $\mathcal{N}$. A move is variant if and only if there exist two P-positions such that the move would connect them if it could be played between them. In "electrical" language: the move would "short-circuit" those P-positions. If there are no such P-positions, the move is invariant.

So the more technical-formal definition that we will need is the following.

Definition 2. A game G is *invariant* if for every position $p \in \mathcal{N}$ there exist $q \in F(p)$such that there are no two $s, t \in \mathcal{P}$ satisfying $p - q = s - t$. The move $p \to q$ is then called *invariant*.

Thus a game is *variant* if it contains an N-position p such that for every $q \in F(p)$ there are $s, t \in \mathcal{P}$ satisfying $p - q = s - t$. Any such move $p \to q$ is called a *variant* move.

Notice that for an invariant game, from every N-position there is an invariant move into a P-position.

Whereas up to now symbols such as p, q, s, t, x, y denoted general k-tuples, we now — until the end of the proof of Theorem 1 — specialize to the case $k = 2$. A general pair of *complementary Beatty sequences* has the form $(A_n, B_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)_{n\geq 1}$, where α, β are positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$. Without loss of generality we may assume $\alpha < \beta$. Then actually $1 < \alpha < 2 < \beta$.

Let $\zeta > 0$ be irrational, $m > n > 0$ integers. Then

$$(m - n)\zeta - 1 < \lfloor m\zeta \rfloor - \lfloor n\zeta \rfloor < (m - n)\zeta + 1.$$

Thus,

$$\lfloor (m - n)\zeta \rfloor \leq \lfloor m\zeta \rfloor - \lfloor n\zeta \rfloor \leq \lfloor (m - n)\zeta \rfloor + 1. \tag{1}$$

Duchêne and Rigo [4] asked the following question, motivated by [3]: given any sequence $S : \mathbb{Z}_{\geq 0} \to \mathbb{Z}^k_{\geq 0}$, is there an invariant game having S as its set of P-positions? They answered this in the negative: consider any sequence $S = (A_n, B_n)_{n\geq 0}$ beginning with $(0, 0), (1, 2), (3, 5), (4, 6)$, such that $\{A_n : n \geq 1\}$ and $\{B_n : n \geq 1\}$ split $\mathbb{Z}_{\geq 1}$. Then from the N-position $(1, 1)$ one must move to $(0, 0)$. Hence $(1, 1)$ is a move. But playing from $(4, 6)$ to $(3, 5)$ is not permitted, since the P-positions constitute an independent set. They put forward the following intriguing

Conjecture 1. *Given any pair of complementary Beatty sequences $S = (A_n, B_n)_{n\geq 1}$, there exists an invariant game having $S \cup \{(0, 0)\}$ as its set of P-positions.*

The bulk of their paper consists of the definition and analysis of a game whose P-positions are complementary Beatty sequences, which they invented to support their conjecture, but the game is of independent interest.

It is our purpose to prove their conjecture. We state this formally:

Theorem 1. *Given any pair of complementary Beatty sequences $S = (A_n, B_n)_{n\geq 1}$, there exists an invariant game G having $S \cup \{(0,0)\}$ as its set of P-positions.*

Proof. Suppose that the assertion is false. Then there exists a position $(a,b) \in \mathcal{N}$ such that for every move x from (a,b), x connects two P-positions. Since $1 < \alpha < 2 < \beta$, the sequences A_n, B_n are increasing. Therefore any move from a *single* pile is invariant. So $a > 0$, $b > 0$, and every variant move x must decrease both piles. Hence for all $(i,j) \in \mathbb{Z}^2_{\geq 1}$ with $i \leq a$, $j \leq b$, there exist $\ell > k \geq 0$ such that $(A_\ell - i, B_\ell - j) = (A_k, B_k)$, or $(A_\ell - i, B_\ell - j) = (B_k, A_k)$. In particular, either
(a1) $(A_\ell - a, B_\ell - b) = (A_k, B_k)$, $a \leq b$ or
(a2) $(A_\ell - a, B_\ell - b) = (B_k, A_k)$, $a \leq b$ or
(b1) $(A_\ell - a, B_\ell - b) = (A_k, B_k)$, $a > b$ or
(b2) $(A_\ell - a, B_\ell - b) = (B_k, A_k)$, $a > b$.

(a) $a \leq b$. Since the pair of Beatty sequences partitions $\mathbb{Z}_{\geq 1}$, we have either $a = B_m$ or $a = A_m$ for some $m \in \mathbb{Z}_{\geq 1}$. If $a = B_m$, then $b \geq a = B_m > A_m$, since $\beta > \alpha$. In particular, $b > A_m$, so from position (a,b) there is the invariant move $b \to A_m$, a contradiction. Therefore $a = A_m$. Also, $m \leq \ell$.

(a1) We have $A_\ell - A_m = A_k$. By (1), $A_{\ell-m} \leq A_k \leq A_{\ell-m} + 1$. Since $1 < \alpha < 2$, we might have $A_{\ell-m} + 1 = A_{\ell-m+1}$. Hence $m \in \{\ell - k, \ell - k + 1\}$. In particular,

$$A_{\ell-m+1} - A_{\ell-m} = 2 \implies k = \ell - m, \tag{2}$$

since $A_{\ell-m+1} - A_{\ell-m} = 2$ implies, by complementarity, $A_{\ell-m} + 1 = B_r$ for some $r \in \mathbb{Z}_{\geq 1}$.

We also have $B_\ell - b = B_k$. Suppose first that $b = B_s$ for some $s \in \mathbb{Z}_{\geq 1}$. Then $B_\ell - B_s = B_k$. By (1), $B_{\ell-s} \leq B_k \leq B_{\ell-s} + 1$. Since $\beta > 2$, $B_{\ell-s} + 1 = A_t$ for suitable $t \in \mathbb{Z}_{\geq 1}$. It follows that $s = \ell - k$.

If $m = \ell - k$, then $s = m$, so $b = B_m$. But then $(a,b) = (A_m, B_m) \in \mathcal{P}$, a contradiction. Hence $m = \ell - k + 1 = s + 1$, $s = m - 1$, $a = A_m$, $b = B_{m-1}$. But then we can use the invariant move $A_m \to A_{m-1}$, a contradiction.

Therefore we must have $b = A_s$. Then $A_s = B_\ell - B_k$. By (1), $B_{\ell-k} \leq A_s \leq B_{\ell-k} + 1$. By disjointness of the Beatty sequences we then have $A_s = B_{\ell-k} + 1$. If $m = \ell - k$, then $b = A_s = B_m + 1$, so there is the invariant move $b \to B_m$. So assume

$$m = \ell - k + 1. \tag{3}$$

Then $b = B_{m-1} + 1$, so

$$(a,b) = (A_m, B_{m-1} + 1).$$

If $A_m - A_{m-1} = 1$, we can move to (A_{m-1}, B_{m-1}) with the invariant move $(1,1)$. So let $A_m - A_{m-1} = 2$. We consider two cases:

(A) $(A_1, B_1) = (1,t)$ $t \geq 3$. Then the first two terms of the sequence $\{A_n\}$ are 1 and 2. Therefore $\alpha < 3/2$, so $\beta > 3$. Hence $B_{n+1} - B_n \geq 3$ for all $n \in \mathbb{Z}_{\geq 1}$. This implies that $(1,2)$ is an invariant move. So we can move $(A_m, B_{m-1}+1) - (2,1) = (A_{m-1}, B_{m-1})$, a contradiction.

(B) $(A_1, B_1) = (1,2)$. Then the moves $(a,b) \to (1,2)$ and $(a,b) \to (2,1)$ are moves from $(a,b) \in \mathcal{N}$ to $(1,2) \in \mathcal{P}$ and $(2,1) \in \mathcal{P}$ respectively. Since we assume that the game is variant, these moves also lead from some P-position to another P-position.

(B1) Consider the move $(a,b) \to (1,2)$. Then for our case (a1), $(A_\ell - (a-1), B_\ell - (b-2)) = (A_\ell - A_m + 1, B_\ell - B_{m-1} + 1) = (A_k, B_k)$. In particular, $B_k - 1 = B_\ell - B_{m-1}$. By (1), $B_{\ell-m+1} \leq B_k - 1 \leq B_{\ell-m+1} + 1$. Equivalently, by (3), $B_k \leq B_k - 1 \leq B_k + 1$, the left hand of which is a contradiction.

(B2) Consider the move $(a,b) \to (2,1)$. Then $(A_\ell - (a-2), B_\ell - (b-1)) = (A_\ell - A_m + 2, B_\ell - B_{m-1}) = (A_k, B_k)$. In particular, $A_k - 2 = A_\ell - A_m$. By (1), $A_{\ell-m} \leq A_k - 2 \leq A_{\ell-m} + 1$. Equivalently, using (3), $A_{\ell-m} \leq A_{\ell-m+1} - 2 \leq A_{\ell-m} + 1$, so $2 \leq A_{\ell-m+1} - A_{\ell-m} \leq 3$. Since $1 < \alpha < 2$, actually $A_{\ell-m+1} - A_{\ell-m} = 2$. By (2) we then have $k = \ell - m$, contradicting (3).

(a2) We have $A_m = A_\ell - B_k$, $b = B_\ell - A_k$. Subtracting,

$$b - A_m = (B_\ell - A_\ell) + (B_k - A_k) \geq B_\ell - A_\ell.$$

Since $\beta > \alpha$, the sequence $\{B_n - A_n\}$ is a nondecreasing function of n. Since $m \leq \ell$ we have $B_\ell - A_\ell \geq B_m - A_m$. Thus $b \geq B_m$. But there cannot be equality, since $(a,b) \in \mathcal{N}$. Therefore $b > B_m$, so there is the invariant move $b \to B_m$.

(b) $a > b$. Analogously to case (a), we deduce that $b = A_m$ for suitable $m \in \mathbb{Z}_{\geq 1}$.

(b1) We have $a = A_\ell - A_k$, $A_m = B_\ell - B_k$. Subtracting, $a - A_m = (B_k - A_k) - (B_\ell - A_\ell) \leq 0$ since $\ell > k$. Thus $a \leq A_m = b$, contradicting $a > b$.

(b2) We have $a = A_\ell - B_k$, $A_m = B_\ell - A_k$. Subtracting, $a - A_m = -(B_k - A_k) - (B_\ell - A_\ell) \leq 0$, contradicting $a > b = A_m$. ■

Possible Future Work.

- Consider invariant games on $r \geq 2$ piles.
- Consider games whose P-positions are given by nonhomogeneous Beatty sequences: $(\lfloor n\alpha + \gamma \rfloor, \lfloor n\beta + \delta \rfloor)$; or $(\lfloor n\alpha_1 + \gamma_1 \rfloor, \ldots, \lfloor n\alpha_k + \gamma_k \rfloor)$. Are they also invariant?
- Invariant games have the property that no move "short-circuits" any P-positions. It may be of interest to consider invariance in a stronger sense: moves that do not short-circuit any two same g-valued positions.

- Is Theorem 1 valid even under the following more stringent definition of invariance? A game G is *invariant* if for every position $p \in \mathcal{N}$, and for every $q \in F(p)$, there are no $s, t \in \mathcal{P}$ satisfying $p - q = s - t$. Perhaps under the even more stringent definition: A game G is *invariant* if for every position p, and for every $q \in F(p)$, there are no $s, t \in \mathcal{P}$ satisfying $p - q = s - t$? A game G is then *variant* if for some position p and some $q \in F(p)$, there are $s, t \in \mathcal{P}$ satisfying $p - q = s - t$. (Here p, q, s, t are general k-tuples, $k \geq 2$.)

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