Ratwyt

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December 28, 2011

In memory of Martin Gardner, who was and remains enchantingly influential and inspiring

Mathematics Subject Classification: 91A46, 91A05, 11B75

Wythoff

In 1907, the Dutch mathematician, Willem Abraham Wythoff [13] invented this game, later vividly explained by Martin Gardner in [7].

Wythoff is played on a pair of nonnegative integers, (M, N). A move consists of either (i) subtracting any positive integer from precisely *one* of M or N such that the result remains nonnegative, or (ii) subtracting the *same* positive integer from both M and N such that the results remain nonnegative. The first player unable to move loses.

Given the position $(3, 3)$, say, the next player wins in a single move: $(3, 3) \to (0, 0)$. The position $(3, 3)$ is called an N-position, because the *Next* player wins. If $M = N = 0$, the next player loses, and the previous player, the one who moved to $(0,0)$, wins. Thus $(0,0)$ is a P-position, because the *Previous* player wins.

If $M > 0$, it is easy to see that $(0, M)$ and (M, M) are N-positions, since the next player can win in one move. On the other hand, $(1,2)$ is a P-position because all its *followers*—positions reached in one move—are N-positions. The first few P-positions are listed in Table 1. Note that every N-position has at least one P-follower, but all followers of a P-position are N-positions. From an N-position, in order to win, a player must move to a P-position. Further, the P- and N-positions partition the set of all game positions: every game position is either a P-position or an N-position but never both..

Table 1: The first few P-positions (A_n, B_n) for Wythoff's game.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
A_n	0	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25	27	29	30	32	33	35	37	38	40	42	43
B_n	0	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41	44	47	49	52	54	57	60	62	65	68	70

The sequences, A_n, and B_n in Table 1, each strictly increasing, have remarkable properties. Note that $B_n = A_n + n$ for all n. But how is A_n computed? A study of the table reveals that for $n \geq 1$, A_n is the smallest positive integer not yet appearing in the sequences. Thus, the next entries in Table 1 are $A_{28} = 45$, $B_{28} = 73$. It follows that the sequences are *complemen-*

tary: every positive integer appears precisely once in these two, recursively defined sequences.

Can a *take-away game*, such as WYTHOFF, be played on the *rational* numbers, rather than the integers? We present such a game here.

A rational take-away game

Given a rational number p/q in lowest terms, a *step* is defined by

$$\frac{p}{q} \to \frac{p-q}{q},$$

if $p/q \geq 1$, otherwise

$$\frac{p}{q} \to \frac{p}{q-p}.$$

The game RATWYT is played on a pair of reduced rational numbers $(p_1/q_1, p_2/q_2)$. A move consists of either (i) doing any positive number of steps to precisely one of the rationals, or (ii) doing the same number of steps to both. The first player unable to play (because both numerators are 0) loses.

For example, from $(3/5, 2/3)$, suppose that Alice moves to $(3/2, 2/3)$. Then Bob can do three steps to both rationals: $(3/2, 2/3) \to (1/2, 2/1) \to (1/1, 1/1) \to (0/1, 0/1)$, and thereby win. Could Alice have made a better initial move? Can Alice win? Suppose she does two steps to each of the initial rationals: $(3/5, 2/3) \to (3/2, 2/1) \to (1/2, 1/1)$. If Bob then moves to $(1/1, 1/1)$, Alice can move to $(0/1, 0/1) = (0, 0)$, winning. We leave it to the reader to verify that in the three remaining possible moves for Bob, Alice can also win in one move. Thus $(3/5, 2/3)$ is an N-position in RATWYT.

Is there a nice winning strategy for RATWYT? If so, what is it?

Continuing with our example, we expand $3/5$ into a continued fraction:

$$3/5 = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}},$$

or $3/5 = [0, 1, 1, 2]$, to use the short notation for continued fractions. Similarly, $2/3 = [0, 1, 2]$.

The sum of the *partial quotients*: $0 + 1 + 1 + 2 = 4$ is called the integer *induced* by the corresponding rational. We claim that playing RATWYT on $(3/5, 2/3)$ is equivalent to playing WYTHOFF on their induced integers, $(3, 4)$!

Supposing this established, then, because $(3, 4)$ is not in Table 1, it is not a P-position, rather an N-position, in WYTHOFF. Hence $(3/5, 2/3)$ is an N-position in RATWYT. So Alice can, in fact, win. Let us re-examine her initial move: $(3/5, 2/3) \to (3/2, 2/3)$. Now $3/2 = [1, 2] \approx 3$ (where $\approx$ denotes the inducing process), so $(3/2, 2/3) \approx (3, 3)$. Alas, $(3, 3)$ is another N-position in WYTHOFF, so Alice missed her opportunity to move to a P-position! Had she made the two-step move $(3/5, 2/3) \to (1/2, 1/1) = ([0, 2], [1]) \approx (1, 2)$, she could have won, since $(1, 2)$ is a P-position in WYTHOFF.

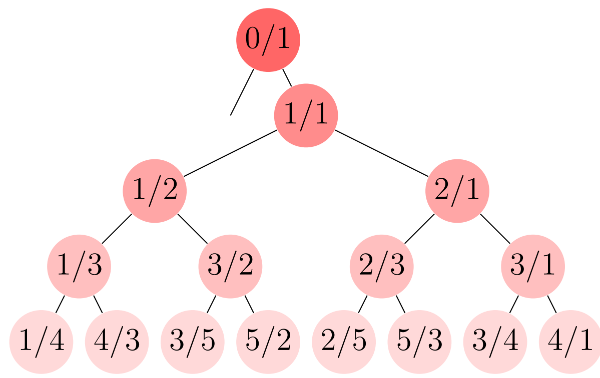

Figure 1: The first five levels of the CW-tree of reduced fractions.

The CW-Tree

To substantiate our claim, that playing RATWYT on $(p_1/q_1, p_2/q_2)$ is equivalent to playing WYTHOFF on their induced integers, we resort to a method of counting the rationals, described in the short, elegant and influential paper of Calkin and Wilf [4].

The Calkin Wilf tree, CW-tree for short, is a binary tree whose nodes are all the nonnegative rational numbers without repetition! The root is the fraction 0/1. The rest of the tree is described inductively by the rule that each vertex i/j has at most two children: a right child $(i+j)/j$; and, if $i > 0$, a left child $i/(i+j)$. The first five levels (levels $0-4$) are drawn in Figure 1. Note that at each step in RATWYT we move towards the root of the tree by one level, so we are guaranteed to eventually reach 1/1 on level 1, and then the root on level 0.

Here are some key properties of the CW-tree.

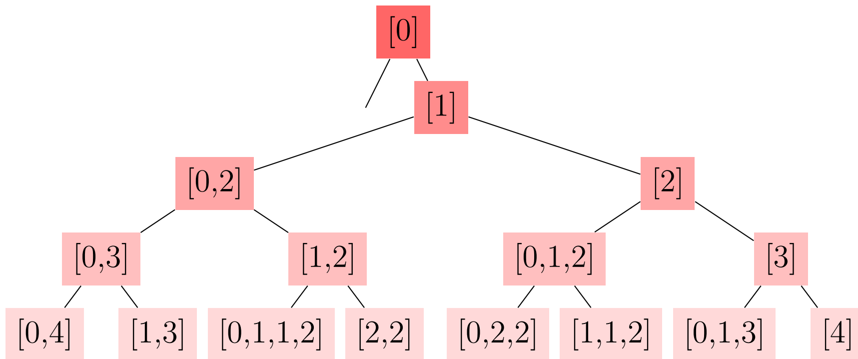

Figure 2: The first five levels of the CW-tree of continued fractions.

1. The numerator and denominator at each vertex are relatively prime: proved by induction on the *level* of the tree.

2. The induced integer of every rational on level k of the tree is k. Indeed, an element r has the right child $r+1$. Incrementing by 1 means incrementing the first term of the continued fraction, hence incrementing the induced integer. The left child is $1/(1+1/r)$. Since taking the reciprocal of a continued fraction amounts to either prefixing or removing a leading 0, neither of which changes the induced integer [8], this also increments the induced integer by one. (See Figure 2.)

3. Every rational number p/q in reduced form occurs precisely once in the CW-tree, namely on level k, where k is the integer induced by p/q. There is a unique path from p/q to the root whose length is precisely k—the same as for every element on level k of the tree, in particular of the rational $k/1 = k$. Hence playing on the rational p/q is equivalent to playing on its induced integer k, as we set out to show.

More games

In [13], Wythoff also discovered that the sequences A_n, B_n can be described explicitly: $A_n = \lfloor n\tau \rfloor$, $B_n = \lfloor n\tau^2 \rfloor$, where $\lfloor x \rfloor$ is the floor function; and $\tau = (1+\sqrt{5})/2$ is the golden ratio. This observation leads to a winning strategy for Wythoff that is polynomial time in the succinct input size $\log(xy)$ of any given game position (x, y).

The preceding section explained how playing Ratwyt on the rationals is equivalent to playing on their induced integers. The same holds for any take-away game. Playing on the integers means restricting travel along the right edge of the CW-tree. Playing on the integer k is equivalent to playing on any of its 2^{k-1} rational siblings on level k of the tree ($k \geq 1$). Here are a few sample take-away games, played on reduced rationals. Keep in mind that the number of steps performed must always preserve non-negativity, and that the first player unable to move loses.

Game I. Let t be a fixed positive integer. A position is a pair (p_1/q_1, p_2/q_2) of rationals. There are two types of moves: (i) do any positive number of steps on precisely one of the rationals, or (ii) do $k > 0$ steps on one of the rationals and $\ell > 0$ on the other, such that $|k - \ell| < t$.

This game, if played on the integers induced by p_1/q_1, p_2/q_2, is a generalization of Wythoff (the case $t = 1$ is Wythoff). The winning strategies given in [5] apply directly to every rational number with the same induced integers. P-positions of generalized Wythoff partition the integers, so

Game I partitions the rationals. Are there any meaningful partitions of the rationals that transcend tree level?

Game II. Let t be fixed positive integer. A position is a single rational, p/q. A move consists of performing up to t steps. The P-positions are all the rationals on nonnegative integer multiples of level $(t+1)$.

Game III. A position consists of m rationals $(p_1/q_1, \ldots, p_m/q_m)$. A move consists of selecting any one of the rationals and performing a positive number of steps. This game amounts to playing Nim on the integers induced by those rationals.

Though playing on the integer k is equivalent to playing on any of its 2^{k-1} rational siblings on level k of the tree ($k \geq 1$), there are games on the rationals with no obvious integer parallels. For example, one may restrict every move to a single tree direction: either move up right, or move up left, both for *impartial* (Wythoff-like) and *partizan* (chess-like) games.

Our game steps amount to single steps in the "slow Euclidean Algorithm". The game Euclid corresponds to playing along the Euclidean Algorithm at a pace chosen by the player. There is a large bibliography on Euclid including [3], [11], [6]. For related games, see [2], [10].

Like the CW-tree, the older Stern-Brocot tree [12], [1] also contains all the reduced rationals. The former has the advantage that the transition from any level to its neighbor is simple; the latter that it is a search tree. Both trees are intimately related to the famous Stern diatomic sequence http://oeis.org/A002487 . A construction of the rationals using recurrence relations is given in [9].

Final remark. In a different context, an anonymous referee recently asked whether there are any "bridges" between combinatorial game theory (CGT) and classical game theory. In the latter, of course, the notion of "rationality" plays a prominent role (in the form of the rational players). Now we see that rationality also plays a role in CGT..

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