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TROMPING GAMES: TILING WITH TROMINOES

Saúl A. Blanco[1]
Department of Mathematics, Cornell University, Ithaca, NY 14850, USA
sabr@math.cornell.edu

Aviezri S. Fraenkel[2]
Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot, Israel
fraenkel@wisdom.weizmann.ac.il

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Abstract

The game of Domineering is a combinatorial game that has been solved for several boards, including the standard 8×8 board. We create new partizan – and some impartial – combinatorial games by using trominoes instead of and along with dominoes. We analyze these Tromping games for some small boards providing a dictionary of values. Further, we prove properties that permit expressing some connected boards as sums of smaller subboards. We also show who can win in Tromping for some boards of the form $m \times n$, for $m = 2, 3, 4, 5$ and infinitely many n.

1. Introduction

1.1. Outline

The game of Domineering was invented by Göran Andersson around 1973, according to [3], [4], and [7]. The two players in Domineering alternately tile a board using a regular domino (a 2×1 tile). The players are usually called *Vertical* (or Left) and *Horizontal* (or Right). They place their tiles, without overlapping, vertically and horizontally, respectively. The player making the last move wins. The game is *partizan*, since the set of moves is different for each player. Conway [4]; and Berlekamp, Conway, and Guy [2] have computed the value of Domineering for several small boards, not necessarily rectangular. D.M. Breuker, J.W.H.M Uiterwijk

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[2]http://www.wisdom.weizmann.ac.il/∼fraenkel

and H.J. van den Herik [3] have determined who wins the game of Domineering for additional boards. In particular, they showed that the first player can win on the classical 8×8 board, which was the original game presented by Andersson.

In this paper we investigate the game of *Tromping*, which is played as Domineering, but with larger tiles instead of, or along with, dominoes, denoted by D for Dominoes. The additional polyominoes are *trominoes*, also called *triominoes*. There are 2 trominoes: a "straight" tromino, i.e., a 3×1 tile, dubbed I-tromino because its shape is that of the letter I; and an L-shaped tromino, i.e., a 2×2 tile with one square removed, dubbed L-tromino.

Any *nonempty* subset of a rectangular board is called a *board* or *subboard* in this paper. The boundaries of its squares are vertical and horizontal only. In Tromping, Vertical and Horizontal place dominoes and I-trominoes vertically and horizontally respectively; the L-tromino can be placed by either player in either one of its four orientations (an "impartial" tromino).

We assume that the reader is familiar with the basic theory of combinatorial games, which can be found in [2], [4] and [1]. A concise 18-page summary with the basic results in combinatorial game theory can be found in [5]. We recall that for a partizan game G, Vertical (Left) can win if $G > 0$, Horizontal (Right) can win if $G < 0$, the second player can win if $G = 0$, and the first player can win if $G \parallel 0$ (G is *fuzzy* (incomparable) with 0). In particular, Horizontal wins as second player if $G \leq 0$, and as first player if $G \lhd\!\!| \, 0$.

In Section 2 we create new partizan combinatorial games, variations of Domineering, by admitting the use of larger tiles. For each game we provide dictionaries of values – where squares that cannot be used during the play (not part of the board) are painted black. We also prove some properties that allow one to express the value of a "connected" board B as the sum of the values of subboards B_1 and B_2. Normally only disjoint subboards form a sum. We point out that all our games are *acyclic* and *short*; that is, there are no repetitions and only finitely many positions.

In Section 3 we discuss some ideas from number theory that can be used to determine the winner of our variations for particular families of boards. These ideas seem to be implicit in Lachmann, Moore, and Rapaport [10].

In Section 4 we provide several results on our variations. For example, we apply the results from number theory discussed in Section 3 to determine the winner for particular boards of the form $m \times n$ for $m = 2, 3, 4, 5$ and infinitely many n. This section contains our main result, Theorem 14, stating that when tiling with either a domino, or an I-tromino or an L-tromino, Horizontal can win on a $3 \times n$ board for every $n > 1$, except that for $n \in \{2, 3, 7\}$, the first player wins. We also include other results about a new iMpartial game, M-Tromping.

Finally, for convenience, whenever no confusion arises, we do not distinguish between the board on which the players are tiling and the actual game.

1.2. Computation of the Values

Values and outcomes of the games were computed by constructing plug-ins for Siegel's *Combinatorial Game Suite* (CGS) [11]. Such plug-ins were needed in order for CGS to understand the rules of the variations of Domineering. No changes were done to the code of CGS. In the computation of the values of the games, CGS considers all possible positions that can be reached in a game without considering efficiency. This approach differs from that of [3] where the authors cut off some positions using an $\alpha-\beta$ search technique to determine the winner.

For the computer experiments, we used a Dell PowerEdge 1750 server with dual 3 GHz Xeon processors and 4 GB of memory running Linux 2.4.21 and a Sun Enterprise 420R server with quad UltraSPARC processors and 4GB of memory running Solaris 8.

For those values too long to write down, we only present the winner, according to whether G is positive, negative, zero, or fuzzy with 0. For example, the value corresponding to the game of I-Tromping (see below) played on the 6×6 board is

$$\pm(\{2|1||1/2\},\{\{6|5||9/2|||2|1||1/2\},\{6|5||9/2|||4|0\}|\{4|0|||-1/2||-1|-2\},\\ \{7/2||3|2|||-1/2||-1|-2\}\}),$$

which is fuzzy with 0, so the first player wins.

Throughout the paper, we use the notations $* = \{0|0\}$, and $n* = n + * = \{n|n\}$ for all integers $n \geq 1$.

2. New Combinatorial Games

2.1. I-Tromping

In our first game, we substitute the domino by an I-tromino. Its rules are exactly the same as for Domineering: the two players, Vertical and Horizontal, tile alternately vertically and horizontally, respectively. Overlapping is not permitted. The player making the last move wins. It might appear that I-Tromping is simply a scaled version of Domineering, but this is not the case for boards that are not rectangular. For square boards there is some resemblance, though: In Domineering, the 2×2 and 3×3 boards both have value ± 1; for I-Tromping the 3×3 and 5×5 boards both have value ± 2.

Figure 1 gives the values of I-Tromping for boards up to six squares, including the 35 boards with 6 squares, excluding their negatives obtained by a right angle turn. Moreover, Figure 2 displays 59 boards with seven squares and their values. Both figures, as well as the rest of figures displaying game values, contain the maximum number of boards that we could fit into four rows.

We enumerate the rows of a board from top down and its columns from left to

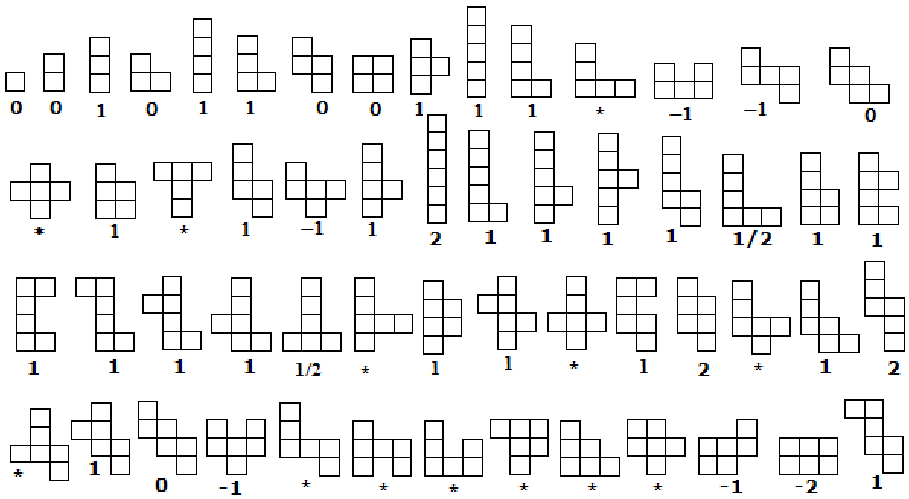

Figure 1: I-Tromping values for boards up to 6 squares.

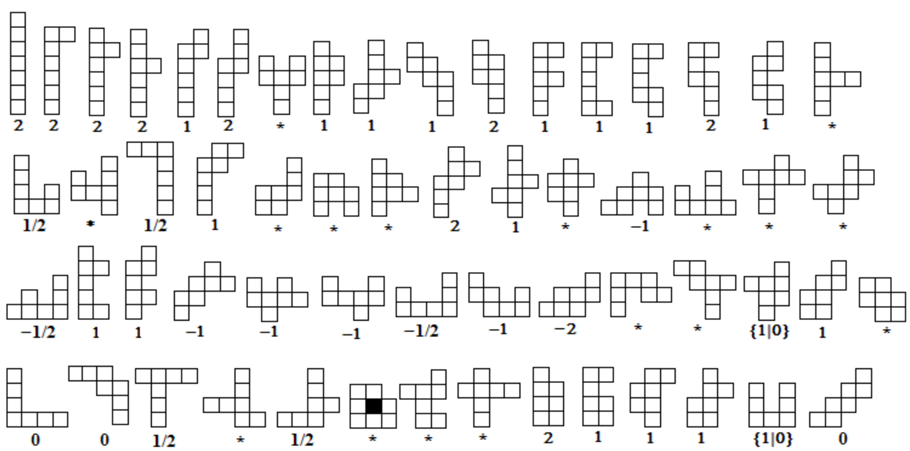

Figure 2: I-Tromping values for some boards with 7 squares.

right, as is common when labeling the rows and columns of a matrix. Since a board need not be rectangular, a row or column can be formed by only one square.

Table 1 depicts the value of I-Tromping for some rectangular boards. We have omitted "messy" values – those that take considerable space to express. Instead, we have used F to indicate that the first player wins; V, that Vertical wins; and H, that Horizontal wins. This notation is also used in the tables (below) containing the values of D-Tromping and L-Tromping (explained below) for small rectangular

	1	2	3	4	5	6
1	0	0	-1	-1	-1	-2
2	0	0	-2	-2	-2	-4
3	1	2	± 2	$\{3 \mid -3/2\}$	$\{4 \mid -1, -1*\}$	$\{4 \mid 0 \mid\mid\mid -1/2 \mid\mid -1 \mid -2\}$
4	1	2	$\{3/2 \mid -3\}$	$\pm 5/2$	$\{3 \mid -2, -2*\}$	$\{3 \mid -3/2 \mid\mid\mid -7/4 \mid\mid -3 \mid -4\}$
5	1	2	$\{1, 1* \mid -4\}$	$\{2, 2* \mid -3\}$	± 2	$\{-3 \mid\mid -3, -3* \mid -8\}$
6	2	4	$\{2 \mid 1 \mid\mid 1/2 \mid\mid\mid 0 \mid -4\}$	$\{4 \mid 3 \mid\mid 7/4 \mid\mid\mid 3/2 \mid -3\}$	$\{8 \mid 3, 3* \mid\mid 3\}$	F

Table 1: Values of I-Tromping for small rectangular boards.

boards.

Definition 1. A board F is said to be *concatenated* to a board G if a domino can be placed horizontally so that its left square tiles a square of F and its right square tiles a square of G. Notation: FG. If the connecting domino is placed in row i of F and row j of G, we also use the notation $FG_{(i,j)}$.

Definition 2. Let $FG_{(i,j)}$ be a concatenation of the board F to G. If each column of $FG_{(i,j)}$ intersects only one of the two boards, then F is (i, j)-*aligned* to G. If F is (i, j)-aligned to G for all possible (i, j), then we simply say that F is *aligned* to G. Otherwise they are *not aligned*.

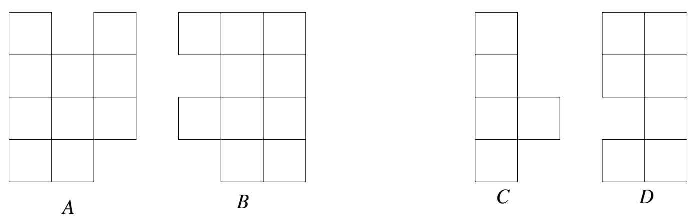

Figure 3: Aligned and non-aligned boards.

Figure 3 depicts an example where A is $(1, 1)$-aligned to B but not $(4, 1)$-aligned, as $AB_{(4,1)}$ will contain in its third column squares from both boards. Also, C is not $(1, 1)$-aligned to D, but it is $(3, 1)$-aligned to D. On the other hand, D is aligned to C.

Definition 3. A rectangular board is called *2-wide* if it has a row with exactly two consecutive squares and no row or column with three consecutive squares.

Clearly any 2-wide board is a 0-game in I-Tromping, since no player is able to move.

These definitions permit us to express some connected boards as sums of their subboards.

Proposition 4. *Let F be aligned to G, and G aligned to H, where G is 2-wide. Then in I-Tromping,*

(a) $FG + GH \leq FGH$,

(b) If $FG = F$ *then* $FGH = F + GH$,

(c) If $GH = H$ *then* $FGH = H + FG$.

Proof. (a) It suffices to show that Horizontal can win as second player in $FG + GH - FGH$ (see Figure 4(a)). If Vertical begins by playing exclusively on F or H, then Horizontal can respond by playing exclusively on $-F$ or $-H$ respectively, and conversely. Since F is aligned to G, Vertical has no moves using squares from *both* of these boards, and the same holds for G and H. The only way Vertical can use squares from two boards is on $-H$ and $-G$ or $-F$ and $-G$. Horizontal can counter these moves by using squares from H and G or F and G, respectively. These are the only options of Vertical since G is 2-wide, so no player can move exclusively on G or move using squares from all three boards. Hence Horizontal can win.

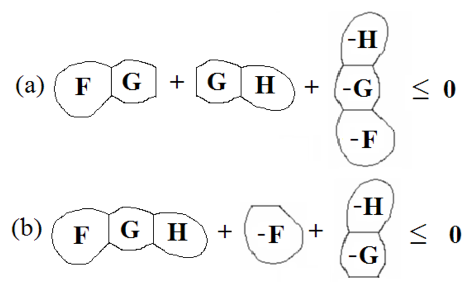

Figure 4: Illustrating the proof of Proposition 4.

(b) We first prove that $FGH \leq F+GH$ by showing that $FGH-F-GH \leq 0$ (see Figure 4(b)). If Vertical, as first player, plays exclusively on F or H, then Horizontal can respond by playing exclusively on $-F$ or $-H$, respectively, and conversely. Further, if Vertical plays using squares from $-H$ and $-G$, then Horizontal can move using squares from H and G, respectively. Since these are the only options for Vertical, Horizontal can win.

To complete the proof, we use the result just proved and part (a):

$$FGH \leq F + GH = FG + GH \leq FGH,$$

and the result follows.

(c) The proof is the same as for (b). $\square$

Corollary 5. *Let F be aligned to $\square\square$, and $\square\square$ aligned to H. Then*

(a) $F\square\square + \square\square H \leq F\square\square H$,

(b) If $F\square\square = F$ *then* $F\square\square H = F + \square\square H$,

(c) If $\square\square H = H$ *then* $F\square\square H = H + F\square\square$.

Proof. These are special cases of Proposition 4 (a) – (c) with $G = \square\square$. $\square$

Remarks. (1) The condition that the boards must be aligned is necessary: Figure 5(a) depicts a case where G is not aligned to H, with G 2-wide. Proposition 4 (a) does not hold, for otherwise $* + \{0|-1\} + 1 = \{1*|*\} \leq 0$, which is false since $\{1*|*\}$ is positive.

(2) The condition that G must be 2-wide is also necessary. Figure 5(b) exhibits a case where $G = 0$ is not 2-wide and the proposition does not hold, since it is easy to verify that

$$\{1|-3\} + \{1|1/2\} = \{2|3/2||-2|-5/2\} \not\leq \{2|3/2||-2|-3\}.$$

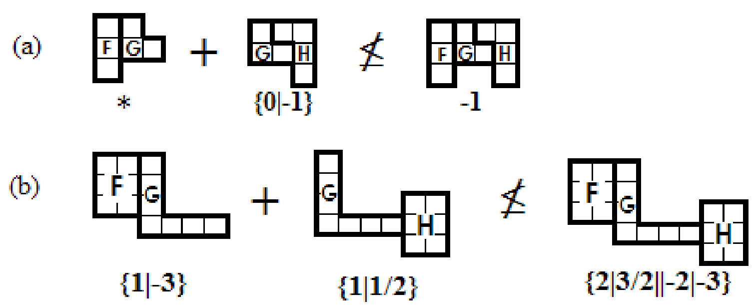

Figure 5: Boards alignment and 2-wideness are necessary.

(3) If $G \neq 0$, then Proposition 4 does not hold. Consider $F = G = H =$ the 3×1 tile. Then $FGH = \pm 2$ (Table 1), and $FG = GH = 2$. Clearly, $2 + 2 \not\leq \pm 2$, so (a) does not hold.

(4) Proposition 4 holds also for playing with a "straight" n-polyomino (an $n \times 1$ tile), if we require G to be $(n-1)$-wide. The proof of this claim is entirely analogous to the above. For increasing n, there is a growing set of $(n-1)$-wide boards G, for each of which Proposition 4 holds. For Domineering, however, Proposition 4 holds if and only if G is the 1×1 tile (see also Proposition 6 below).

The last remark shows that for increasing size of the (smallest) tiling polyomino, the power of Proposition 4 increases, as it permits to express as sums a growing variety of boards that are not disjoint. This can already be observed for I-Tromping.

In Figure 6 we have applied Proposition 4(b) to two different cases: The first one with G being the 2×2 board, F the 3×3 board, and H the 3×1 tile. In the second case, we take G to be a 2×2 board with one square removed. Note that G need not be rectangular. From Figure 6 we see that

$$\{2|1||-2|-3\} = \pm 2 + \{0|-1\}$$
$$\pm(2)^* = \pm 2 + *$$

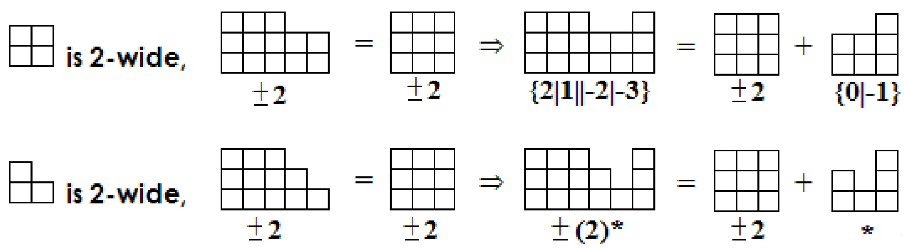

Figure 6: Two sample applications of Proposition 4 (b).

In general we cannot divide a "connected" board into pieces so that the value of the original board equals the sum of the values of the smaller boards, but here we can.

2.2. D-Tromping

Here Vertical and Horizontal alternate in tiling with either a domino or an I-tromino. The player making the last move wins. A dictionary of values for boards of up to six squares is depicted in Figure 7. Table 2 presents some of the values of rectangular boards.

Proposition 6. *Let F be aligned to $\Box$, and $\Box$ aligned to H. Then for* D-*Tromping we have,*

(a) $F\Box + \Box H \leq F\Box H$,

(b) *If* $F\Box = F$ *then* $F\Box H = F + \Box H$,

(c) *If* $\Box H = H$ *then* $F\Box H = H + F\Box$.

Proof. Same as Proposition 4 with G replaced by $\Box$. $\square$

This proposition holds also for Domineering; see [4] Ch. 10.

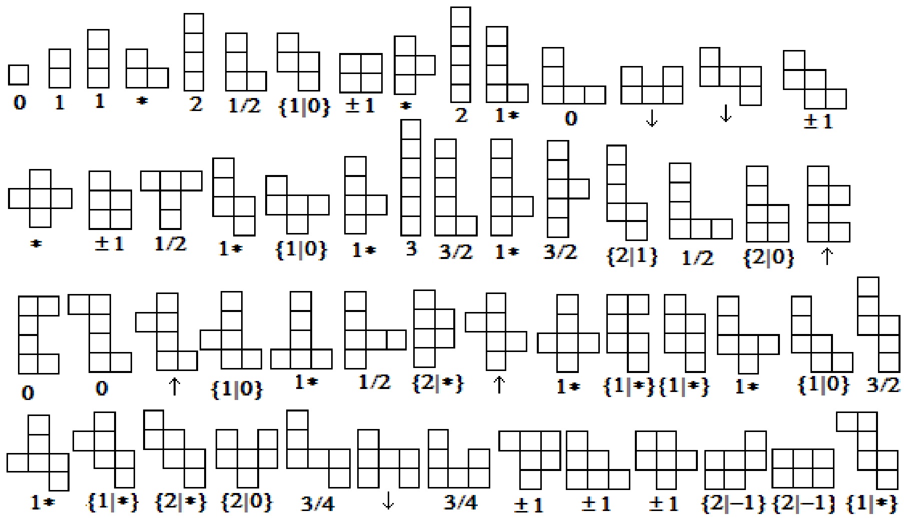

Figure 7: D-Tromping values for boards up to 6 squares

	1	2	3	4	5
1	0	-1	-1	-2	-2
2	1	± 1	$\{2\mid -1\}$	$\{2\mid 0\|\mid -1*\}$	$\{0, \{3\mid 0\}\mid -1, \{0\mid -2\}\}$
3	1	$\{1\mid -2\}$	± 2	$\{2\mid -1\|\mid -2\uparrow\}$	$\{-1, \{3\mid -1\}\mid -7/4, \{-1\mid -4\}\}$
4	2	$\{1*\|\mid 0\mid -2\}$	$\{2\downarrow\|\mid 1\mid -2\}$	$\pm 1*$	F
5	2	$\{1, \{2\mid 0\}\mid 0, \{0\mid -3\}\}$	$\{7/4, \{4\mid 1\}\mid 1, \{1\mid -3\}\}$	F	F

Table 2: Values of D-Tromping for small rectangular boards

2.3. L-Tromping

In this game we tile with an L-tromino in addition to the I-tromino and domino. The L-tromino adds a total of 4 new moves to the set of moves of each player, since it can be rotated and placed in 4 different positions on a rectangular board. A dictionary of values of this game for small boards is exhibited in Figures 8 and 9, and Table 3 depicts the values of L-Tromping for some rectangular boards.

	1	2	3	4	5
1	0	-1	-1	-2	-2
2	1	± 1	$\{2\mid -1\}$	$\{1*, \{2\mid 0\}\mid -1*\}$	$\{0, \{3\mid 0\}\mid 0, \{0\mid -2\}\}$
3	1	$\{1\mid -2\}$	± 2	H	H
4	2	$\{1*\mid \{0\mid -2\}, -1*\}$	V	F	F
5	2	$\{0, \{2\mid 0\}\mid 0, \{0\mid -3\}\}$	V	F	F

Table 3: Values of L-Tromping for small rectangular boards.

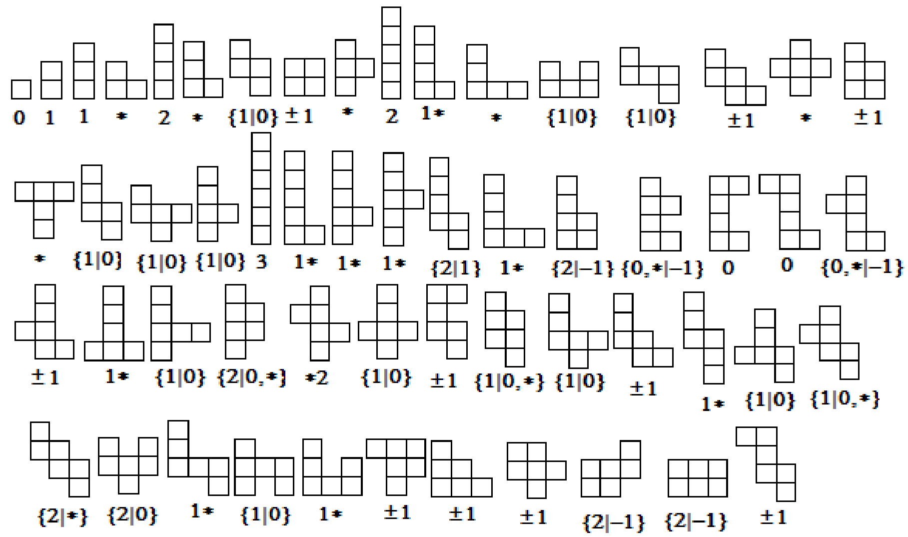

Figure 8: L-Tromping values for boards up to 6 squares.

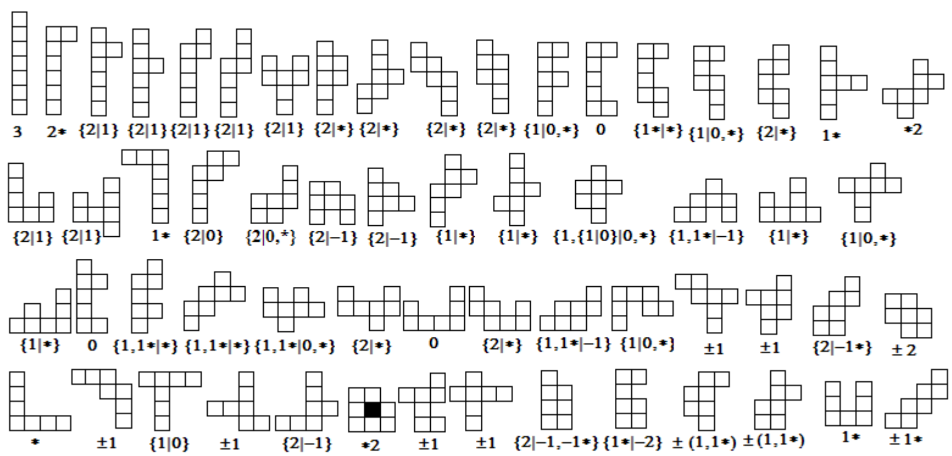

Figure 9: L-Tromping values for some boards with 7 squares

As can be seen in the dictionaries of Figures 8 and 9, the value *2 is attained several times on small boards of L-Tromping. The first time is on a board of only 6 squares. For domineering, on the other hand, it is not so easy to construct a board with value *2. Such a board was recently constructed by G.C. Drummond-Cole [6]. It appears that the values of L-Tromping are hotter than those of our preceding games. If this is indeed the case in general, it may be due to the L-tromino, which can be used by both players. Thus the game resembles more an impartial game in which every nonzero value is hot.

L-Tromping also satisfies Proposition 6 of D-Tromping.

Here is another property of L-Tromping:

Lemma 7. *Let B be a board and let B' be a subboard obtained from B by removing either a single $k \times 1$ subboard S of B, where $k \in \{2, 3\}$ or an* L*-tromino. Then in* L*-Tromping, $B < 0$ implies $B' \not> 0$.*

Proof. Suppose that Vertical begins tiling on B with a 2×1 or 3×1 tile or with an L-tromino, so that B' results. If $B' > 0$, then Vertical can win as player I on B, so $B \mid\mid 0$ or $B > 0$, a contradiction. Hence $B' \not> 0$. □

Remark 8. An analogous statement holds for Domineering, I-tromping, D-Tromping, by removing a 2×1, 3×1, $k \times 1$ ($k \in \{2, 3\}$) subboard, respectively.

3. Connection with the Frobenius Problem

Definition 9. Let $n \in \mathbb{Z}_{>0}$, and $A \subset \mathbb{Z}_{>0}$ a finite set. We say that A is a *nonnegative basis* for $\mathbb{Z}_{>n}$ if every integer greater than n can be written as a linear nonnegative integer combination of elements of A.

Example. $A = \{4, 5, 6\}$. One can easily see that A is a nonnegative basis for $\mathbb{Z}_{>7}$. Indeed, any nonnegative integer combination of elements of A has the form $4k_1 + 5k_2 + 6k_3$ for $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$, not all of them zero: Considering the positive integers modulo 4 we get

- For n of the form $4k, k > 0$, we use $k_2 = k_3 = 0$.
- For n of the form $4k + 1, k > 0$, we use $k_1 = k - 1,\ k_2 = 1,\ k_3 = 0$.
- For n of the form $4k + 2, k > 0$, we use $k_1 = k - 1,\ k_2 = 0,\ k_3 = 1$.
- For n of the form $4k + 3, k > 1$, we use $k_1 = k - 2,\ k_2 = 1,\ k_3 = 1$.

It is clear that there can be more than one nonnegative basis for $\mathbb{Z}_{>n}$, $n \in \mathbb{Z}_{>0}$. For example, consider $\mathcal{B}_1 = \{12, \ldots, 23\}$ and $\mathcal{B}_2 = \{4, 9, 11, 14\}$. The set $\mathcal{B}_1$ is a nonnegative basis for $\mathbb{Z}_{>11}$ by Proposition 10 below. Furthermore, we can see that $\mathcal{B}_2$ is also a nonnegative basis for $\mathbb{Z}_{>11}$ (we can follow the preceding example, noting that $\mathcal{B}_2$ is a complete system of remainders modulo 4). In fact, since $11 \in \mathcal{B}_2$, this set is actually a nonnegative basis for $\mathbb{Z}_{\geq 11}$.

The following is a result on nonnegative bases.

Proposition 10. *If $k \in \mathbb{Z}_{>0}$, then the set $A = \{k, k+1, \ldots, 2k-1\}$ forms a nonnegative basis for $\mathbb{Z}_{>k-1}$.*

Proof. For any $n \in \mathbb{Z}_{\geq k}$, n can be written uniquely in the form $n = qk + r$, with $q \in \mathbb{Z}_{\geq 0}$ and $k \leq r < 2k$, so $r \in A$. □

For example, the set $\{3, 4, 5\}$ forms a nonnegative basis for $\mathbb{Z}_{>2}$.

While playing a game, a board B will normally be divided into smaller boards. For example, in I-Tromping, Vertical can divide a $3 \times n$ board into two as soon as he performs his first move. On the other hand, Horizontal also has the power of dividing a board, by avoiding certain moves. For example, if the game takes place on a 3×8 board, Horizontal can divide such a board into two 3×4 subboards by not tiling squares from the fourth and fifth column in the same move.

The above observation shows that, when playing I-Tromping or D-Tromping, if Horizontal can divide an $m \times n$ board into subboards in which he wins, the whole board will be won by Horizontal.

More concretely, fix $m \in \mathbb{Z}_{>0}$ and let $A = \{n_1, n_2, \ldots, n_k\} \subset \mathbb{Z}_{>0}$ so that Horizontal wins I-Tromping or D-Tromping on the $m \times n_i$ board, $i = 1, \ldots, k$. Then Horizontal wins on the $m \times n$ board, for $n = \sum_{i=1}^{k} a_i n_i$ and $a_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, k$.

As an interesting connection with number theory, the largest integer N for which there is no solution to the Diophantine equation $N = \sum_{i=1}^{k} x_i n_i$ in nonnegative integers x_i is known as the *Frobenius number* for A. See [9], **C7**. Finding the Frobenius number for a given set of positive integers is known as the *Frobenius problem* (sometimes referred to as *Frobenius coin problem* or *coin problem*).

So in particular, if we can find a nonnegative basis $\mathcal{B}$ for $\mathbb{Z}_{\geq k}$ with each element $n_i \in \mathcal{B}$ representing a subboard $m \times n_i$ (for fixed m) that can be won by Horizontal, then Horizontal will win all $m \times n$ boards, where $n \geq k$. This is the use of nonnegative bases in our study.

The above application relies on the following theorem, which is a generalization of a result implicit in [10].

Theorem 11. *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a set of boards such that F_i is aligned to F_j for all $i, j \in \{1, \ldots, k\}$, and $F_i < 0$ for all i. If G is constructed by concatenating any finite number of copies of boards from $\mathcal{F}$, not necessarily distinct copies, then, for* I*-Tromping and* D*-Tromping, $G < 0$.*

Proof. The idea is to notice that Horizontal has the power to divide G into smaller boards by refraining to tile more than one subboard. The following is a winning strategy for Horizontal: Divide G into subboards and play exclusively on a given copy of a game, without ever tiling across the border of 2 adjacent subboards. Vertical is unable to play on more than one copy by alignment. In this way, Horizontal establishes vertical boundaries, which cannot be crossed by Vertical. Since $F_i < 0$, for $i = 1, \ldots, k$, Horizontal can force a win on each subboard. In this fashion, G becomes the sum of the copies of elements of $\mathcal{F}$, and so Horizontal can win G. $\square$

Note that this method cannot be applied directly to L-Tromping, as each player is allowed to use an L-tromino, and so Horizontal no longer has the advantage of restricting the play to a given copy. Does Theorem 11 still hold anyway? Theorem 14 below shows it does for a special case.

4. Who Wins on Horizontal Strips?

4.1. Results for Particular Boards

We will focus on games of the form $m \times n$, where $m = 2, 3, 4, 5$. The tables were computed using the plug-in for the CGS software.

As mentioned earlier, we denote in the tables a win for Horizontal by H, a win for Vertical by V, and a win for the first player by F.

We apply the following method: For fixed m, we try to find several values n_i for which the games played on the $m \times n_i$ board are negative. Then by Theorem 11, concatenating any number of those boards will produce a negative game. If the n_is also form a nonnegative basis for $\mathbb{Z}_{>k}$ for some $k \in \mathbb{Z}_{>0}$, then we can determine the winner for a game played on an $m \times n$ board, $n > k$.

Note that Horizontal wins I-Tromping on $2 \times n$ boards for all $n \geq 3$, as Vertical cannot move at all on any such board. For $n \in \{1, 2\}$, the first player loses. Of course all results for boards of the form $m \times n$ have a similar version for boards of the form $n \times m$, with the respective changes. More specifically, the board of the form $n \times m$ is the *negative* of the $m \times n$ board. If a board is a win for Horizontal, its negative is a win for vertical, and vice-versa. If a board is a win for the first (second) player, its negative is also a win for the first (second) player, respectively.

4.2. I-Tromping for $3 \times n$ Boards

Many values found so far are fuzzy, as seen in Table 4. More information is needed to determine the winner for an arbitrary n. However, since Horizontal wins in the 3×6 game, Theorem 11 implies that Horizontal wins for all boards of the form $3 \times 6k$ for $k \in \mathbb{Z}_{\geq 1}$.

Remark 12. If we concatenate (in an aligned manner) a board A that is won by Horizontal and a board B that is won by the first player we obtain a board that is won by either Horizontal or the first player. This observation is made in [10], Table (2-3). Indeed if Horizontal plays first, he can start in B and counter Vertical's moves in A. For instance, since $19 = 6 + 13$, we can conclude the winner of I-Tromping on the 3×19 board is either Horizontal or the first player, for Horizontal wins the

3×6 board and the first player wins the 3×13 board. Similarly, from Table 4 we conclude that either the first player or Vertical wins I-Tromping on the $3 \times m$ board, where

$$m \in \{3+6j, 4+6j, 5+6j, 7+6k, 9+6k, 10+6k, 11+6k, 13+6l, 14+6l, \\ 15+6l, 16+6l, 17+6l \mid j,k,l \in \mathbb{Z}_{\geq 3}, k,l \in \mathbb{Z}_{\geq 2}\}.$$

Remark 13. Notice that we cannot tell who wins in the concatenation of a board that is won by Vertical and one that is won by the first player since Horizontal can cross the vertical boundaries of these two boards.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	V	V	F	F	F	H	F	V	F	F	F	H	F	F	F	F	F	H

Table 4: I-Tromping values for $3 \times n$ boards

4.3. I-Tromping for $4 \times n$ Boards

We find that the boards for $n \in \{6, \ldots, 11\}$ have negative values. Further, $\{6, \ldots, 11\}$ forms a nonnegative basis for $\mathbb{Z}_{>5}$ by Proposition 10. Hence, Horizontal wins I-Tromping on $4 \times n$ boards for all $n \geq 6$, by Theorem 11. Table 5 shows the winner for $n < 6$.

n	1	2	3	4	5
	V	V	F	F	F

Table 5: I-Tromping values for $4 \times n$ boards

4.4. I-Tromping for $5 \times n$ Boards

By Table 6, H can win for all $n \in \{6, \ldots, 11\}$, so Proposition 10 implies that Horizontal can win all games with $n \geq 6$. The table also shows who wins for $n < 6$.

n	1	2	3	4	5	6	7	8	9	10	11
	V	V	F	F	F	H	H	H	H	H	H

Table 6: I-Tromping values for $5 \times n$ boards

4.5. D-Tromping for $2 \times n$ Boards

By computation, $2 \times n$ boards have negative values for $n \in \{4, 9, 11, 14\}$. This set forms a nonnegative basis for $\mathbb{Z}_{\geq 11}$. Table 7 shows the values for boards of width less than 11. Therefore, we know who wins D-Tromping on boards of the form $2 \times n$ for all $n \geq 1$.

n	1	2	3	4	5	6	7	8	9	10
	V	F	F	H	F	F	F	H	H	F

Table 7: D-Tromping values for $2 \times n$ boards

4.6. D-Tromping for $3 \times n$ Boards

The game has negative values for $n \in \{4, 5, 6\}$. Hence by the example in Section 3 Horizontal also wins the game for $n > 7$. Table 8 gives the values for $n \leq 7$. So the winner of D-Tromping on boards of the form $3 \times n$ is known for all $n \geq 1$.

n	1	2	3	4	5	6	7
	V	F	F	H	H	H	F

Table 8: D-Tromping values for $3 \times n$ boards

4.7. D-Tromping for $4 \times n$ Boards

Table 9 summarizes the results we have obtained so far. Since Horizontal wins the 4×6 board, Theorem 11 implies that Horizontal also wins on all boards of the form $4 \times 6k,\ k \in \mathbb{Z}_{>0}$. Furthermore, Remark 12 in Section 4.2 implies that either Horizontal or the first player is the winner of D-Tromping for boards of the form $4 \times m$, where $m \in \{4 + 6k, 5 + 6k, 7 + 6k \mid k \in \mathbb{Z}_{\geq 1}\}$.

n	1	2	3	4	5	6	7
	V	V	V	F	F	H	F

Table 9: D-Tromping values for $4 \times n$ boards

4.8. L-Tromping for $2 \times n$ Boards

Table 10 shows who can win for $1 \leq n \leq 14$. Since the first player wins for $1 < n \leq 14$, we are wondering whether the the first player wins L-Tromping on the $2 \times n$ board for all $n > 1$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	V	F	F	F	F	F	F	F	F	F	F	F	F	F

Table 10: L-Tromping values for $2 \times n$ boards

4.9. L-Tromping for $3 \times n$ Boards

Table 11 summarizes the winner of L-Tromping for $3 \times n$ boards, $n \leq 12$. This table, along with Theorem 14 below, gives us the winner of L-Tromping for $3 \times n$ boards, for every $n \in \mathbb{Z}_{>0}$.

n	1	2	3	4	5	6	7	8	9	10	11	12
	V	F	F	H	H	H	F	H	H	H	H	H

Table 11: L-Tromping values for $3 \times n$ boards

Although we do not know whether Theorem 11 holds for L-Tromping, the theorem holds for $3 \times n$ boards. The key here is the geometry of the board. Horizontal has the upper hand by tiling squares in the second row, by which he reserves some squares in the first and third row for future moves.

Theorem 14. *Horizontal can win* L-*Tromping both as first and as second player on every* $3 \times n$ *board for all* $n > 1$*, except that for* $n \in \{2, 3, 7\}$*, the first player can win.*

We use the following notation: For a $3 \times n$ board B, we call n the *width* of B, and write $w(B) = n$. The following definition is used in the proof of Theorem 14.

Definition 15. The *vertical part* (*horizontal part*) of an L-tromino L is the vertical (horizontal) domino that is contained in L.

Proof of Theorem 14. We proceed by induction on n. Since Table 11 contains the information for $n \leq 12$, we assume that $n \geq 13$. Specifically, for an arbitrary but fixed integer $n \geq 13$, we assume that every $3 \times m$ board is a win for Horizontal for every $1 < m < n$, except that for $m \in \{2, 3, 7\}$ the first player wins. We will show that Horizontal wins also on an $3 \times n$ board. There are two cases:

(1) `Horizontal starts.` We think of B as being divided into two pieces: a subboard A of size 3×4 on the left and an abutting subboard $B' = B \setminus A$ of size $3 \times m$ to its right, where $m = n - 4 \geq 9$.

It is straightforward to verify that $A(0, 3, 0) \leq 0$ and $A(1, 3, 0) \leq 0$, where the numbers indicate the squares removed at the extreme right or left end of A, here the rightmost end of A. Thus $A(1, 3, 0)$ is A without the top rightmost and three middle rightmost squares. Horizontal's overall strategy is to begin tiling A, where he can win, and to counter Vertical's moves in whatever part Vertical chooses to move. Horizontal can win in B' by induction.

Horizontal starts by moving on A with an I-tromino, covering the squares in the second row adjacent to B' (the rightmost three squares of the middle row), resulting in $A(0, 3, 0)$. If Vertical does cross the boundary with an L-tromino, then Horizontal responds in B' with Horizontal's winning strategy there. If Vertical moves in A, then Horizontal can use his strategy on $A(0,3,0)$ or $A(1,3,0)$ to win what is left of A. So after his first move, Horizontal has transformed the board into two "basically disjoint" negative pieces, in the sense that any move that crosses the boundary between both pieces has its vertical part in B', transforming $A(0,3,0)$ into $A(1,3,0)$.

Since $B' < 0$ by induction and $A(0,3,0) \leq 0$, $A(1,3,0) \leq 0$, Horizontal wins B by always responding in the part in which Vertical tiled at least two adjacent squares.

(2) `Vertical starts.` Vertical's first move can be one of three kinds.

(a) An I-tromino I. Notice that placing I in either the first or last column makes $B < 0$ by induction, since now the play is on the $3 \times (n-1)$ board. So we can assume that Vertical divides B into two nonempty disjoint subboards B_1 and B_2 of size $3 \times k_1$ and $3 \times k_2$ respectively, where $k_1 + k_2 + 1 = n$. If $k_1 = k_2 = 6$ (so $n = 13$), then $B_1 < 0$ and $B_2 < 0$ by Table 11. We can thus assume that either $k_1 \geq 7$ or $k_2 \geq 7$. So we assume, without loss of generality, that $k_2 \geq 7$. Moreover, if $k_1 \leq 3$ then $9 \leq k_2 < n$, and so $B_2 < 0$ by induction. To show that $B < 0$, it suffices to show that Horizontal can win as first player on $B \setminus I$, that is, $B \setminus I \lhd\!| \, 0$.

(i) If $k_1 = 1$, then $B_1 = 1$ (the game value 1); and $B_2 < 0$ by induction. Horizontal's response is to tile the 3×3 board adjacent to I in its middle row. Let B_3 denote the 3×5 subboard that contains B_1 and the two I-trominoes, and let $B_4 = B \setminus B_3$. In B_4 Horizontal can win by induction, since $n \geq 13$, implies $w(B_4) \geq 8$. Even if Vertical crosses the boundary covering the rightmost upper or lower square of B_3 with an L-tromino whose vertical part is in B_4, Horizontal can clearly win in B_3. In fact, $B_3 < 0$, since Horizontal has two moves for the single move available to Vertical. So the following is a winning strategy for Horizontal as second player on $B_3 + B_4$: Counter Vertical's moves on B_3 and B_4 respectively, wherever Vertical chooses to play.

(ii) If $k_1 \in \{2,3\}$, then $B_1 \mid\mid 0$ and $B_2 < 0$ by induction. So Horizontal wins $B_1 + B_2$ as first player by moving first on B_1 and countering Vertical's moves on B_2.

(iii) If $k_1 \geq 4$, $k_1 \neq 7$ and $k_2 \neq 7$, then $B_1 < 0$ and $B_2 < 0$ by induction. In particular, Horizontal wins $B_1 + B_2$ as first player, that is, $B \setminus I \lhd\!| \, 0$.

(iv) For $k_1 = k_2 = 7$, CGS shows that $B_1 + B_2 < 0$, so $B \setminus I \lhd\!| \, 0$. Letting $G = B_1 = B_2$, Table 11 shows that $G \mid\mid 0$. Yet CGS implies further that $G < 1$ (Vertical has less then one move advantage over Horizontal in $G - 1$) and $G + 3 \mid\mid 0$, so even if Vertical would start in $G + G$, Horizontal would have an advantage of at least two moves over Vertical.

(v) If $k_1 \geq 4$ and $k_1 \neq 7$ and $k_2 = 7$, then $B_1 < 0$ by induction, and $B_2 \mid\mid 0$. So Horizontal wins $B_1 + B_2$ by playing first on B_2. Thus $B \setminus I \lhd\!| \, 0$.

(vi) If $k_1 = 7$ and $k_2 \neq 7$, then $B_1 \mid\mid 0$ and $B_2 < 0$ by induction. So Horizontal wins $B_1 + B_2$ by playing first on B_1. Hence $B \setminus I \lhd\!| \, 0$.

These are all the possible values of k_1, so we established that $B < 0$ for this case.

(b) A domino D. Placing D on the first or last column, we have $B < 0$ by induction, since then the game can be thought of as that on the $3 \times (n-1)$ board. Indeed, let S be the remaining square in the column where D is placed. Horizontal can pretend that Vertical placed an I-tromino instead of a domino, and use Horizontal's winning strategy as in (a). If Vertical covers S, he must use an L-tromino whose vertical part is on the $3 \times (n-1)$ board, so its effect is equivalent to tiling a vertical domino.

Otherwise D is placed so as to form nonempty subboards B_1 and B_2 as in (a). Notice that if the moves are completely contained in B_1 and B_2, (a) shows that $B \setminus D \lhd\!\!| \; 0$, so Horizontal does not need to cover S in order to be victorious. Furthermore, If Vertical covers S, he must use an L-tromino whose vertical part is contained in either B_1 or B_2, and thus its effect would be equivalent to tiling with a vertical domino in either B_1 or B_2, respectively. Thus $B < 0$ in this case.

(c) An L-tromino L. There are two possibilities for the horizontal part of L: it either lies on the first or third row of B, or it lies on the second row of B.

Suppose first that the horizontal part of L lies on the first or third row. We divide B into two rectangular subboards B_1 and B_2 so that B_2 contains L completely and B_1, abutting B_2, shares its vertical boundary with the outside vertical part of L. The board B and its division are depicted in Figure 10. Notice that the division of B depends on the orientation of L, as can be seen in Figure 10(a) and Figure 10(b).

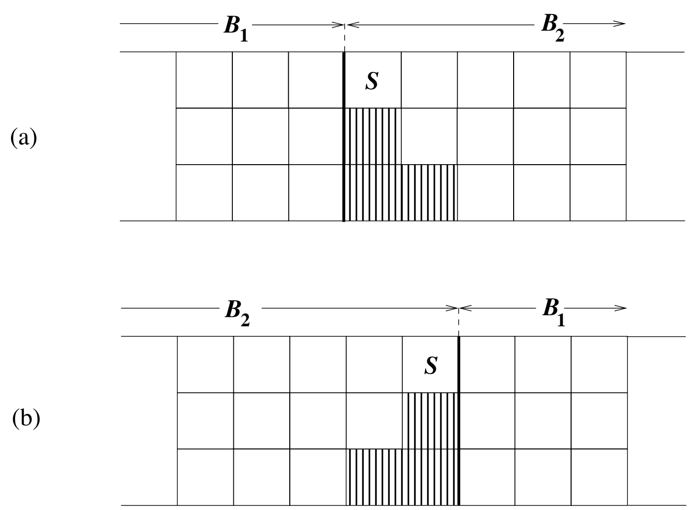

Figure 10: B_2 and B_1 share the vertical boundary of the vertical part of L. By definition, B_1 and B_2 are rectangular boards and B_2 contains L. The respective positions of B_1 and B_2 change depending on the orientation; for instance, (a) shows B_1 to the left of B_2 while (b) shows B_2 to the left of B_1. However, by symmetry, we can assume that B_1 is to the left of B_2.

By symmetry, we can assume, without loss of generality, that L covers squares at positions $(1,k),(1,k+1),(2,k)$, rather than at $(2,k)$, $(3,k)$, $(3,k+1)$, for some $1 \leq k < n$. For instance, the L-tromino depicted in Figure 10(a) can be considered to cover squares $(1,k),(1,k+1),(2,k)$ by flipping B so that the first row becomes the third row. We can further assume that B_1 is located to the left of B_2.

Let C denote the 3×2 rectangle containing L. We may assume that B_1 is nonempty since otherwise, after Vertical placed L, Horizontal can place another L-tromino L^H, tiling C completely, whence the game is played on the $3 \times (n-2)$ board, which Horizontal can win by induction. So $w(B_1) \geq 1$. The same argument shows that we may assume $w(B_2) \geq 3$.

Assume first that $w(B_1) = 1$. Then a winning move for Horizontal is to tile an I-tromino on squares $(2,3),(2,4),(2,5)$ of B. Let B_1' be the 3×5 rectangle consisting of the first five columns of B that contain the two trominoes tiled so far, $B' = B \setminus B_1'$. An easy analysis shows that $B_1' < 0$, even if Vertical chipped away one of the squares (1,5) or (3,5) with an L-tromino on B'. Moreover, notice that $w(B') \geq 13 - 5 = 8$, and so Horizontal wins B' by induction. Thus $B \setminus L \lhd\!\!| \; 0$.

Now suppose that $w(B_1) \geq 2$. Analogously to the notation introduced in part (1) above, denote by $B_2(0,1,2)$ the subboard B_2 without the leftmost middle and two bottom leftmost squares, that is, B_2 after it has been tiled with L. By induction and Table 11 (since $w(B_2) \geq 3$), B_1 and B_2 are either fuzzy or negative. We consider all four possibilities.

(i) Suppose $B_1 < 0$ and $B_2 < 0$. Let S be the square in column k that L did not cover (see Figure 10). Horizontal first counters Vertical's L-tromino with his winning strategy as first player on $B_2(0,1,2)$, and then counters Vertical's moves in whatever part Vertical chooses to play. However, Vertical might cross the borders by tiling S with an L-tromino whose vertical part is on B_1. Then Horizontal can counter this move with his winning strategy on B_1. But Vertical has also tiled the square S on $B_2(0,1,2)$. It thus suffices to show that Horizontal's strategy on $B_2(0,1,2)$ does not require to tile S.

We show that this is the case by considering all possible values for $w(B_2)$. Since $B_2 < 0$, we may assume $w(B_2) \geq 4$. Also $w(B_2) \neq 7$ since $B_2 < 0$. If $w(B_2) \in \{5,6\}$, it is easy to verify that S is not required in Horizontal's winning strategy as first player on $B_2(0,1,2)$. Indeed, tiling $(2,k+1),(2,k+2),(2,k+3)$ (recall that L covers the squares $(1,k),(1,k+1),(2,k)$, for some $1 < k < n$) is a winning move for Horizontal on $B_2(0,1,2)$. If $w(B_2) \in \{8,9\}$, Horizontal can win by tiling a domino D on squares $(2,k+1),(2,k+2)$. Let C_1 be the board formed by the three leftmost columns of $B_2(0,1,2)$. Horizontal's D is a winning move on C_1 and on $C_1(1,0,0)$, and $C_1(0,0,1)$, the boards obtained from C_1 by removing the last square in the first and third row respectively. Furthermore, the board $E = B_2(0,1,2) \setminus C_1$ has width $w(E) = w(B_2) - 3 \in \{5,6\}$, so $E < 0$. Thus after playing D, the game becomes $B_1 + C_1 + E$, which is negative since each of its components is, so Horizontal wins. The same holds if C_1 is replaced by $C_1(1,0,0)$ or $C_1(0,0,1)$. If $w(B_2) \geq 10$, then Horizontal's winning move is to tile L^H, the L-tromino that along with L covers columns k and $k+1$. This move transforms the game into a disjoint sum $B_1 + (B_2 \setminus \{L, L^H\})$, which is negative by induction, since $w(B_2 \setminus \{L, L^H\}) \geq 8$.

(ii) Suppose $B_1 \mid\mid 0$ and $B_2 \mid\mid 0$. Then $w(B_1) = w(B_2) = 7$, since we are assuming that $n > 12$ and $B_1 \mid\mid 0$ implies $w(B_1) \in \{2,3,7\}$ by induction. In this case, Horizontal can play an L-tromino L^H so that L and L^H cover two columns of B completely. After placing L^H, the game is played on two disjoint boards of width 7 and 5, respectively. CGS shows that the sum of these games is negative, and thus $B \setminus L \lhd\!\mid 0$.

(iii) Suppose $B_1 < 0$ and $B_2 \mid\mid 0$. Hence $w(B_2) \in \{2,3,7\}$ by induction. If $w(B_2) \in \{2,7\}$, then Horizontal's tiling L^H (where L^H is as before) is a winning move. Indeed, after tiling L^H, the game has value $B_1 + B'$, where B' is a $3 \times \ell$ board with $\ell \in \{0,5\}$. Therefore $B_1 + B' < 0$, so Horizontal wins. If $w(B_2) = 3$, then Horizontal's winning move is to tile $(2, n-1), (2, n)$ (the two rightmost middle squares of B_2) with a domino. After this move Vertical can only move on $B_1 < 0$ (possibly also tiling S, which still leaves a move for Horizontal on B_2). Thus $B \setminus L \lhd\!\mid 0$.

(iv) Suppose $B_1 \mid\mid 0$ and $B_2 < 0$. Then $B_1 \in \{2,3,7\}$ by induction. Horizontal's strategy is to tile the I-tromino I covering $(2, k+1), (2, k+2), (2, k+3)$, as depicted in Figure 11. Let B_3 be the 3×4 subboard containing I and L, and let $B_4 = B_2 \setminus B_3$. Notice that any move of Vertical on $B \setminus \{L, I\}$ is a move on either B_4 or B_1, possibly also tiling one of the squares S, A, C (Figure 11). Clearly Horizontal wins B_3 by at least one move. Also, $w(B_4) \geq 13 - 11 = 2$. For the case $w(B_1) = w(B_4) = 3$, CGS shows that $B_1 + B_4 = 0$. But then clearly $B = B_1 + B_3 + B_4 < 0$, so Horizontal wins. For any other combination of $B_1 \in \{2,3,7\}$, $B_4 \in \{2,3,7\}$, CGS shows that $B_1 + B_4 < 0$, so Horizontal again wins. Hence we may assume that $w(B_4) \notin \{2,3,7\}$. Thus by induction, $B_4 < 0$.

It is Vertical's turn to play on $B \setminus \{L, I\}$, and he can win as first player on B_1. We show that Horizontal's I enables to save enough moves to counter this advantage. If $w(B_1) = 7$, Vertical is not able to save any extra moves on B_1 since $B_1 - 1 < 0$ (see (a)(iv) above). Thus Horizontal can counter Vertical's move on B_1 with moves on B_1 and the move saved on B_3. If $w(B_1) \in \{2,3\}$, then one easily sees that the winning moves for Vertical as first player do not involve tiling S. Furthermore, if Vertical does not use a winning move as first move on B_1, Horizontal is able to guarantee his victory on B_1 quite easily, and thus on the whole game, since $B_4 < 0$. So we can assume that Vertical's first move on B_1 is a winning move, and in particular does not involve S. Furthermore, notice that Vertical can cover at most one of the squares A or C, and so by tiling I, Horizontal has actually saved two moves if S is not tiled. If $w(B_1) = 2$, these two moves are enough to counter Vertical's advantage on B_1.

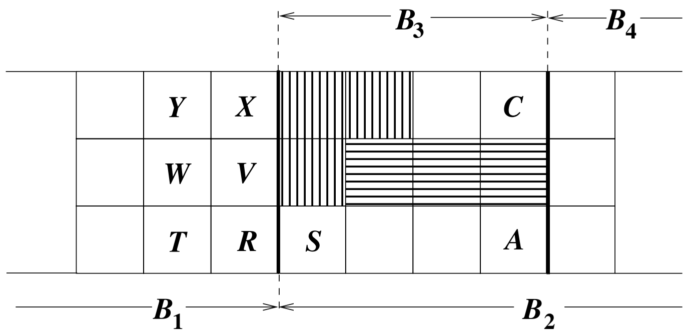

Figure 11: Depictions of the subboards used in case (iv). Notice that $w(B_1) \in \{2, 3, 7\}$ since $B_1 \mid\mid 0$.

It remains to see what happens if $w(B_1) = 3$. The only winning move for Vertical as first player on B_1 that tiles R is tiling R, T, W with an L-tromino. If Vertical utilizes this move, then Horizontal counters by tiling V, X, Y or by an I-tromino in the top row of B_1. Then Vertical has one extra move on B_1's leftmost column. This extra move can be countered with the two extra moves that Horizontal saved on B_1. On the other hand, if Vertical's winning move does not tile R, then Horizontal can counter Vertical's winning move on B_1 with the L-tromino S, R, V. Then Vertical has only one extra move left on B_1, and such extra move can be countered with the move that Horizontal saved on B_1 (in this case only one horizontal move is guaranteed to be saved on B_1, since Horizontal tiled S). Thus Horizontal wins.

Thus Horizontal wins if the horizontal part of L lies in the first or third row.

Now assume that the horizontal part of L lies in the second row of B. As in (c) above, we let C denote the 3×2 rectangle containing L. Then C partitions the board B into two subboards B_1, B_2 of width k_1, k_2 respectively, where $k_1 + k_2 + 2 = n$. If B_1 is empty, then Horizontal wins by induction, since $w(B_2) \geq 11$, and any move of Vertical on C is equivalent to a domino on B_2, here and below. The same holds if B_2 is empty. We may thus assume that both are nonempty, and proceed very much like in (a)(i)-(vi) above. Specifically,

(i) $k_1 = 1$. We use the strategy of (a)(i), where the I-tromino I of width 1 is replaced by C of width 2. Horizontal's I-tromino is now placed in positions (2,4), (2,5), (2,6), so we have only $w(B_4) \geq 7$ (rather than 8 in (a)(i)). If $w(B_4) \geq 8$, then the argument of (a)(i) applies. For $w(B_4) = 7$, CGS shows that B partially tiled with C and Horizontal's I-tromino, is negative. Thus $B \setminus L \lhd\!\mid 0$.

The cases (a)(ii)-(vi) carry over to the present case, except that Vertical's initial I is replaced by C. Thus $B < 0$, since Horizontal wins B as both first and second player. $\square$

4.10. Results on M-Tromping

Motivated by L-Tromping, we create a game in which each of the two players is only allowed to use L-trominoes. Since the sets of moves for each player are identical, M-Tromping is an i**M**partial game. The *Sprague-Grundy theory* tells us that its values are *nimbers* $*a$, $a \in \mathbb{Z}_{\geq 0}$. So for each game of M-Tromping G, we have,

$$G = \{*a_1, *a_2, \cdots, *a_n | *a_1, *a_2, \cdots, *a_n\} = *a, \quad a := \operatorname{mex}\{a_1, a_2, \cdots, a_n\},$$

where for any set $S \subsetneq \mathbb{Z}_{\geq 0}$, $\operatorname{mex} S = \min \mathbb{Z}_{\geq 0} \setminus S =$ smallest nonnegative integer not in S.

We present the value of M-Tromping for small rectangular boards in Table 12, where we denote each nimber $*b$ by b to simplify notation. Note that such a table is symmetric, since every nimber is its own negative.

$m \backslash n$	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	1	2	0	3	1
3	0	2	0	1	2	2
4	0	0	1	0	1	0
5	0	3	2	1	0	1
6	0	1	2	0	1	1

Table 12: Sprague-Grundy values for M-Tromping played on the $m \times n$ board

Let $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$. Denote by $G(m, n)$ the game of M-Tromping played on an $m \times n$ rectangular board. Furthermore, let $G_1(m, n)$ be the game of M-Tromping played on an $m \times n$ board with two opposite corners removed (say, the upper-left and lower-right corner) and let $G_2(m, n)$ be the game of M-Tromping played on an $m \times n$ board with the first and last square of the first row removed. It is convenient to define $G(m, 0) = G_1(m, 0) = G_1(m, 1) = G_2(m, 0) = G_2(m, 1) = 0$. We prove,

Theorem 16. *The games $G(2, n)$, $G_1(2, n + 1)$, and $G_2(2, n + 1)$ have the same value for all $n \in \mathbb{Z}_{\geq 0}$.*

Before proving Theorem 16, we need some notation to describe moves on $G(2, n)$, $G_1(2, n + 1)$, and $G_2(2, n + 1)$. Label the squares of both rows of $G(2, n)$ and $G_1(2, n + 1)$ from left to right with numbers from $[n] = \{1, 2, \ldots, n\}$ in increasing order. Similarly, label the squares in the first row of $G_2(2, n+1)$ with $\{2, 3, 4, \ldots, n\}$ in increasing order from left to right and the squares in the second row with $[n + 1]$ increasingly from left to right (see Figures 12 and 13.) In any of these boards let $(a, a + 1; a + 1)$ represent the move where the L-Triomino is covering the squares labeled $a, a+1$ from the first row and $a+1$ from the second one, with $a, a+1 \in [n]$.

Define $(a, a+1; a), (a; a, a+1), (a+1; a, a+1)$ similarly, where the semicolon is used to separate the labels from different rows.

Proof of Theorem 16. It is enough to show that $G(2, n) + G_1(2, n+1) = 0$ and $G(2, n) + G_2(2, n+1) = 0$, for $n > 0$.

We proceed by induction on n. For $n = 0, 1$, all three games have value 0 and for $n = 2$, all have value $*$. For $n > 2$, suppose that for $m < n$, $G(2, m)+G_1(2, m+1) = 0$ and $G(2, m) + G_2(2, m + 1) = 0$.

The rules below, depicted in Figure 12, guarantee a win for the second player in $G(2, n) + G_1(2, n + 1)$: If the first player's first move is

(1) $(k; k, k + 1)$ in $G(2, n)$ then move $(k; k, k + 1)$ in $G_1(2, n + 1)$, for $1 \leq k < n$.

(2) $(k, k + 1; k)$ in $G(2, n)$ then move $(k; k, k + 1)$ in $G_1(2, n + 1)$, for $1 \leq k < n$.

(3) $(k - 1, k; k)$ in $G(2, n)$ then move $(k - 1, k; k)$ in $G_1(2, n + 1)$, for $1 < k \leq n$.

(4) $(k; k - 1, k)$ in $G(2, n)$ then move $(k - 1, k; k)$ in $G_1(2, n + 1)$, for $1 < k \leq n$.

(5) $(k; k, k + 1)$ in $G_1(2, n + 1)$ then move $(k; k, k + 1)$ in $G(2, n)$, for $1 \leq k < n$.

(6) $(k - 1, k; k)$ in $G_1(2, n + 1)$ then move $(k - 1, k; k)$ in $G(2, n)$, for $1 < k \leq n$.

Now each player has moved exactly once, and the two boards have been divided into at most four subboards: two rectangular boards T_1 and T_2 with a corner square removed, one $G(2, m)$, and one $G_1(2, m + 1)$ for some $0 \leq m < n$ (these last two could be empty). By the induction hypothesis, $G(2, m)+G_1(2, m+1) = 0$. Since T_1 and T_2 have the same shape (after rotating and flipping if necessary), $T_1 + T_2 = 0$. So after the first move, the second player can turn the game into a zero game.

The rules for the remaining possible first moves of the first player in $G(2, n) + G_1(2, n + 1)$ now follow:

(7) $(k-1, k; k+1)$ in $G_1(2, n+1)$ then move $(k, k+1; k)$ in $G(2, n)$, for $1 < k < n$.

(8) $(k-1; k, k+1)$ in $G_1(2, n+1)$ then move $(k; k-1, k)$ in $G(2, n)$, for $1 < k < n$.

Now each player has moved exactly once, and the two boards have been divided into at most four subboards: two rectangular boards T_1 and T_2 with a corner square removed, one $G(2, m)$, and one $G_2(2, m + 1)$ for some $0 \leq m < n$ (these last two could be empty). By the induction hypothesis, $G(2, m) + G_2(2, m + 1) = 0$. As above, T_1 and T_2 have the same shape, hence $T_1 + T_2 = 0$. So after the first move, the second player can turn the game into a zero game.

Thus $G(2, n) + G_1(2, n + 1) = 0$.

The rules below, depicted in Figure 13, guarantee a win for the second player in $G(2, n) + G_2(2, n + 1)$: If the first player's first move is

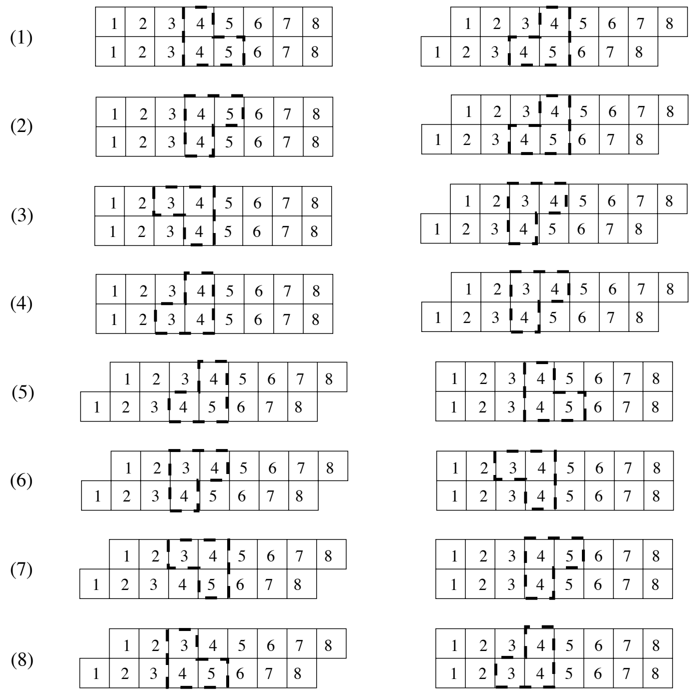

Figure 12: Second player's response in the game $G(2,8)+G_1(2,9)$ according to the rules (1)-(8) with $k=4$. The first player has 8 possible first moves, one for each pair of the 4 orientations of the L-tromino and the 2 boards G, G_1. First player's moves are depicted in the first column of boards; second player's response in the second column.

(1') $(k; k, k+1)$ in $G(2,n)$ then move $(k+1; k, k+1)$ in $G_2(2, n+1)$, for $1 \leq k < n$.

(2') $(k, k+1; k)$ in $G(2,n)$ then move $(k+1; k, k+1)$ in $G_2(2, n+1)$, for $1 \leq k < n$.

(3') $(k-1, k; k)$ in $G(2,n)$ then move $(k; k, k+1)$ in $G_2(2, n+1)$, for $1 < k \leq n$.

(4') $(k; k-1, k)$ in $G(2,n)$ then move $(k; k, k+1)$ in $G_2(2, n+1)$, for $1 < k \leq n$.

(5') $(k+1; k, k+1)$ in $G_2(2, n+1)$ then move $(k; k, k+1)$ in $G(2,n)$, for $1 \leq k < n$.

(6') $(k; k, k+1)$ in $G_2(2, n+1)$ then move $(k; k-1, k)$ in $G(2,n)$, for $1 < k \leq n$.

Now, each player has moved exactly once, and the two boards have been divided into at most 4 subboards: two rectangular boards T_1 and T_2 with a square removed, one $G(2,m)$, and one $G_1(2, m+1)$ for some $0 \leq m < n$ (these last two could be empty). By the inductive hypothesis, $G(2,m)+G_1(2,m+1)=0$ and since T_1 and T_2 have the same shape (after rotating and flipping if necessary), then $T_1+T_2=0$.

So after the first move, the second player can guarantee that the game becomes a zero game.

The rules for the remaining possible first moves of the first player in $G(2,n)+G_2(2,n+1)$ now follow:

(7') $(k,k+1;k)$ in $G_2(2,n+1)$ then move $(k-1,k;k)$ in $G(2,n)$, for $1<k\leq n$.

(8') $(k-1,k;k)$ in $G_2(2,n+1)$ then move $(k-1,k;k-1)$ in $G(2,n)$, for $2<k\leq n$.

Each player has now moved exactly once, and the two boards have been divided into at most four subboards: two rectangular boards T_1 and T_2 with a square removed, one $G(2,m)$, and one $G_2(2,m+1)$ for some $0\leq m<n$ (these last two could be empty). By the inductive hypothesis, $G(2,m)+G_2(2,m+1)=0$ and since T_1 and T_2 have the same shape (after rotating and flipping if necessary), then $T_1+T_2=0$. So after the first move, the second player can guarantee that the game becomes a zero game.

Thus $G(2,n)+G_2(2,n+1)=0$. This completes the proof. □

As noted above, M-Tromping is an impartial game and therefore the winner of this game is either the first or second player. For example, $G(2,3)$ is a first-player win, and $G(2,4)$ is a second-player win. Curiously, however, by just removing a corner from a $2\times n$ board, M-Tromping becomes a first-player game for all $n>1$. We prove this in the following theorem.

Theorem 17. *For $n>1$, let $B(n)$ be the value of M-Tromping played on a $2\times n$ board with one corner removed. Then $B(n)\,||\,0$ for all n.*

Proof. Without loss of generality, we assume that the square removed is the lower-right corner. After labeling both columns of the board with numbers from $[n]$ in increasing order, we note that the statement is trivial for $n=2$. For $n>2$, we recognize two cases,

***n* is odd.** The first player moves $(\frac{n+1}{2};\frac{n+1}{2}-1,\frac{n+1}{2})$, this will bisect the board into two pieces with value $B(\frac{n+1}{2}-1)$. After this move, the value of the game is $B(\frac{n+1}{2}-1)+B(\frac{n+1}{2}-1)$, which is a zero-game, as $B(\frac{n+1}{2}-1)$ is its one negative. Therefore the first player (who is the second to play on $B(\frac{n+1}{2}-1)+B(\frac{n+1}{2}-1)$) can win the game.

***n* is even.** The first player moves $(\frac{n}{2},\frac{n}{2}+1;\frac{n}{2})$. This will split the original board into two pieces: $G(2,\frac{n}{2}-1)$ and $G_1(2,\frac{n}{2})$, respectively. By Theorem 16, these two pieces have the same value when playing M-Tromping, and so its sum is 0. Hence the first player (who is the second to play on $G(2,\frac{n}{2}-1)+G_1(2,\frac{n}{2})$) can win the game. □

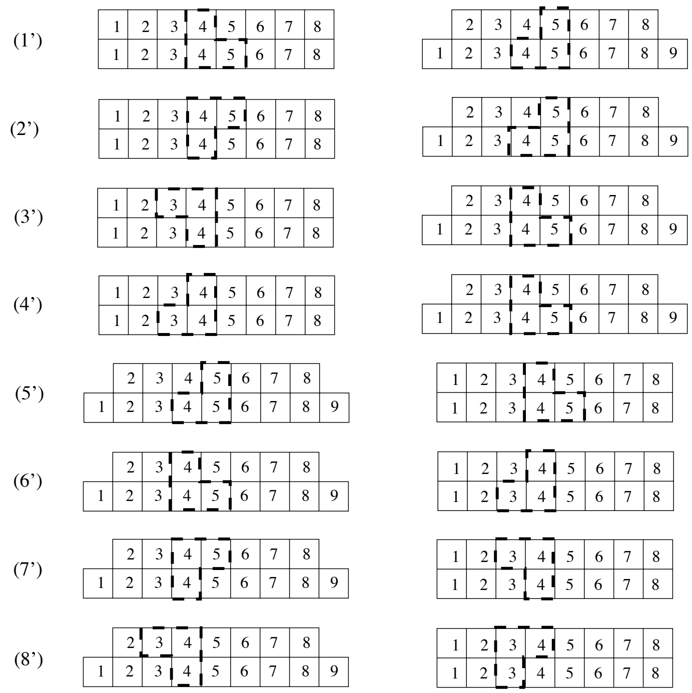

Figure 13: Second player's response in the game $G(2,8)+G_2(2,9)$ according to the rules (1')-(8') with $k=4$. The first player has 8 possible first moves, one for each pair of the 4 orientations of the L-tromino and the 2 boards G, G_2. First player's moves are depicted in the first column of boards; second player's response in the second column.

4.11. Some Families of Values

When playing I-Tromping or D-Tromping on square boards, notice that the sets of moves for Horizontal and Vertical are negatives of one another. So these games are either a first or second-player win. Notice that the winner of both I-Tromping and D-Tromping on the 5×5 board is the first player. In contrast, it is shown in [4] page 116 that Domineering on a 5×5 board is a second player win. We also note that Tables 1 and 2 are antisymmetric, since the value of a game on a board B is the negative of the game on board B', where B' is the rotation of B by $90°$ about its center (clockwise or counterclockwise).

Finally, in addition to the dictionaries of values already provided, we present in Figure 14 some patterns of boards that have a clearly discernible pattern of values. One can prove the validity of some of these patterns by using the properties that were pointed out in Propositions 4 and 6. For example, to prove by induction the validity of the first sequence of I-Tromping , we use Proposition 4 (a) and (b), and the well-known fact that $* + * + \cdots + *$ is 0 if we add an even number of stars, and $*$ if we add an odd number of them.

The last two lines of Figure 14 constitute a sequence of values for L-Tromping. The values obtained are,

$$*, \downarrow, \underbrace{\{\downarrow \mid -1\}, -1, -1^*, -1\downarrow}, \underbrace{\{-1\downarrow \mid -2\}, -2, -2^*, -2\downarrow}, \underbrace{\{-2\downarrow \mid -3\}, -3, -3^*, -3\downarrow} \cdots$$

The emerging pattern is a four-term block of the form $\{(1-a)\downarrow|-a\}, -a, -a^*, -a\downarrow$ for $a \in \mathbb{Z}_{\geq 1}$ increasing in steps of 1.

We remark, in passing, that the "double cross" in Fig 14 (second board from left in first row of L-Tromping) has value $\{\{0|-1\}, *|*, 0, \{0|-1\}\} = \{*|0, \{0|-1\}\}$ (by domination) $= \{*|0\}$ (by reversibility) $= \downarrow$. Note that, perhaps a bit counterintuitively, the best opening move for Horizontal is to tile with a domino the middle 2 squares.

5. Concluding Remarks

- We proved some properties of I-Tromping (Proposition 4) which generalize those of Domineering (Proposition 6). This may lead one to think that there could be some isomorphism between I-Tromping and Domineering. We believe, however, that this is not the case. In particular, we propose that there is no position in I-Tromping with value

$$\pm(0, \{\{2|0\}, 2+_2 |\{2|0\}, -_2\}),$$

  which is the value of a 4×4 board in Domineering.

- It is natural to generalize the results of this paper to larger polyominoes. Results of the form of Proposition 4 grow stronger with increasing size of the smallest participating polyomino.

- The games presented here were analyzed for *normal* play; that is, the player making the last move wins. One can also analyze these games for *misère* play, where the player making the last move loses, but then the usefulness of sums is lost.

- For some of the boards, we need more experimental computations. But, since we are dealing with a very particular kind of board (horizontal strips), perhaps some heuristic techniques can be used.

- We do not know whether Theorem 11 can be applied in general for L-Tromping, although for the particular cases we have analyzed here, it seems plausible. One approach would be to generalize Theorem 14.

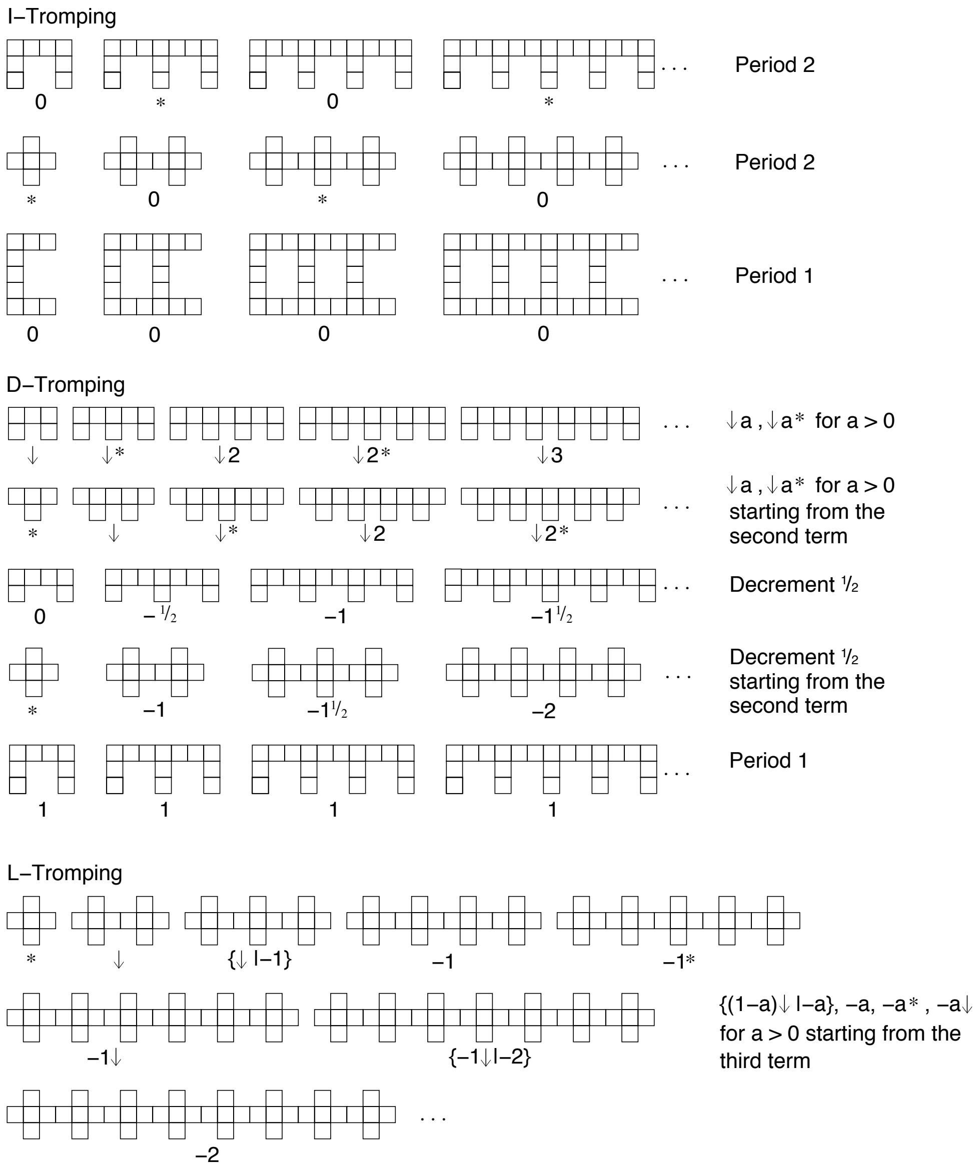

Figure 14: Families of patterns.

- M-Tromping on $2 \times m$ boards has interesting properties: while removing two corners basically does not change the game (see Theorem 16 for a precise statement), removing a single corner makes the game fuzzy.

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