Extensions of Wythoff’s Game

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Abstract

We determine the maximal set of moves for 2-pile take-away games with prescribed $P$-positions $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$ for $n \in \mathbb{Z}_{\geq 1}$ where $\alpha \in (1, 2)$ is irrational, $1/\alpha + 1/\beta = 1$. This was done in [3] for the special case $\alpha = \text{golden ratio}$. We generalize the infinite Fibonacci word to an infinite word $W$ with alphabet $\Sigma = \{a, b\}$, in which $\alpha$ replaces the golden ratio, and we analyze the set \{\[s \in \mathbb{Z}_{\geq 0} : W(s) = b, W(s + x) = a \}\} for any fixed value of $x$.

1 Introduction

Generalized Wythoff (see [5]) is a two-player game, played on two piles of tokens. The two possible types of moves are: a. remove a positive amount of tokens from one pile, b. remove $k > 0$ tokens from one pile and $\ell > 0$ from the other, provided that $|k - \ell| < t$, where $t \in \mathbb{Z}_{\geq 1}$ is a parameter of the game. The player making the last move wins.

The case $t = 1$, in which the second type of move is to remove the same amount of tokens from both piles, is the classical Wythoff game [11], a modification of the game Nim. From among the extensive literature on Wythoff’s game we mention just three: [2], [5], [12].

Since the game is finite, every position of the game is either an $N$-position – a position from which the Next player can win, or a $P$-position – a position from which the Previous player can win. The game positions are encoded in the form $(x, y)$, where $x$, $y$ are the sizes of the piles and $x \leq y$. It was shown in [5] that the set of $P$-position, $\mathcal{P}$, for generalized Wythoff is \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0} \}$, where $\alpha = [1; t, t, t, \ldots] = (2 - t + \sqrt{t^2 + 4})/2$ and
\( \beta = 1 + 1/(\alpha - 1) \). Notice that the condition \( \beta = 1 + 1/(\alpha - 1) \) is equivalent to \( 1/\alpha + 1/\beta = 1 \); and when \( \alpha = [1; t, t, t, \ldots] \), then \( \beta = \alpha + t \).

We consider two games to be identical if they have the same set of \( P \)-positions. Let

\[
\alpha^{-1} + \beta^{-1} = 1, \quad \alpha \text{ irrational}, \quad 0 < \alpha < \beta. \tag{1}
\]

Then \( 1 < \alpha < 2 < \beta \). In this paper we seek the largest set of moves in games whose \( P \)-positions are \( \{([n\alpha], [n\beta])\}_{n \geq 0} \). The existence of such a game for an arbitrary irrational \( \alpha \) was proven in [8].

For example, [4] describes a nice set of moves for \( \alpha = [1; 1, t, 1, t, \ldots] = 1 + (\sqrt{t^2 + 4t} - t)/2 \): A player can (a) remove a positive amount of tokens from one pile or (b) remove the same amount of tokens, \( k \), from both piles as long as \( k \notin \{2, 4, \ldots, 2t - 2\} \) or (c) remove \( 2t + 1 \) tokens from one pile and \( 2t + 2 \) tokens from the other.

It turns out that the largest set of moves is \( \mathbb{V} \setminus \mathcal{M} \) where \( \mathbb{V} \) is the set of all moves consisting of either taking \( x > 0 \) from a single pile, or else taking \( x > 0, y > 0 \) from both; and \( \mathcal{M} \) is the set of moves that allow the players to move from one \( P \)-position to another.

We will consider the set of \( y \)'s such that \((x, y) \in \mathcal{M}\) for any fixed \( x \). It turns out that there is a strong relation between this set and a generalized version of the Fibonacci word, \( \mathcal{W} \). In fact, we will have to investigate the set of \( y \)'s such that \( \mathcal{W}(y) = b \) and \( \mathcal{W}(y + x) = a \).

This analysis can be done using a generalization of the Fibonacci numeration system (for information on numeration systems, see [6]), and also using techniques from the theory of words and morphisms of words. In this paper we chose the latter approach.

\section{Preliminaries}

An \textit{invariant} game is a game for which the moves are playable from any position (see [4]). A \textit{symmetric invariant} game is a game where the piles are unordered.

We consider symmetric invariant take-away games, played on two piles of tokens. We denote a position of the game by a pair \((a, b)\) such that \( a \leq b \). A move is also denoted by a pair \((x, y)\) such that \( x \leq y \). Notice that there can be two ways of playing this move from the position \((a, b)\): to \((a - x, b - y)\) or to \((a - y, b - x)\) (we may need to change the order if \( a - x > b - y \)).
We assume throughout, without stating so explicitly, that we can never take away from any pile more than the pile size.

The set of moves $V$ defined in the introduction can be written as $V = \{(x, y) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : x \leq y, \ y \neq 0\}$. For any subset of moves $\mathcal{V} \subseteq V$, let $\mathcal{P}(\mathcal{V})$ denote the set of $P$-positions of the game defined by $\mathcal{V}$ (the $P$- and $N$-positions of a game are defined in the introduction).

For example, for Generalized Wythoff,

$$\mathcal{V} = \{(0, k) : k \in \mathbb{Z}_{\geq 1}\} \cup \{(k, \ell) : k, \ell \in \mathbb{Z}_{\geq 1}, \ 0 \leq \ell - k < t\},$$

where $\alpha = [1; t, t, t, \ldots]$ and $1/\alpha + 1/\beta = 1$.

Note that the definition of $P$- and $N$-positions implies that from a $P$-position the players can move only to $N$-positions and from an $N$-position there exists a move to a $P$-position. In particular, there is no move from any $P$-position to any other $P$-position. We say that the set $\mathcal{P}$ of $P$-positions of any given game constitute an independent set.

It was shown in [8], that for any irrational $\alpha \in (1, 2)$, there exists an invariant game with a set of moves, $\mathcal{V}$, such that $\mathcal{P}(\mathcal{V}) = \{([\alpha n], [\beta n]) : n \in \mathbb{Z}_{\geq 0}\}$, where $\alpha, \beta$ satisfy (1). Notice that (1) implies that $\{[\alpha n] : n \in \mathbb{Z}_{\geq 1}\}$, $\{[\beta n] : n \in \mathbb{Z}_{\geq 1}\}$ are a pair of complementary Beatty sequences (see [1], [5]).

In this paper we study the following question: Fix an irrational $\alpha \in (1, 2)$. What is the maximal set of moves $\mathcal{V} \subseteq V$ such that

$$\mathcal{P}(\mathcal{V}) = \{([\alpha n], [\beta n]) : n \in \mathbb{Z}_{\geq 0}\},$$

where $\beta = 1 + 1/(\alpha - 1)$?

**Proposition 1.** Let $\mathcal{M} \subseteq V$ be the subset of moves that allow the players to move from one $P$-position to another. The maximal set of moves, $\mathcal{V}_{\text{max}}$, that satisfies (3) is $V \setminus \mathcal{M}$.

**Proof.** Since $\mathcal{P}$ is an independent set, $\mathcal{M} \cap \mathcal{V} = \emptyset$ for every subset of moves $\mathcal{V}$ that satisfies (3). So $\mathcal{V} \subseteq V \setminus \mathcal{M}$.

Take a set $\mathcal{V}_0$ that satisfies (3). The existence of an invariant game $G$ with move set $\mathcal{V}_0$ satisfying (3) was proven in [8]. In particular, in $G$ the move set $\mathcal{V}_0 \subseteq V \setminus \mathcal{M}$ permits to move from every $N$-position into a $P$-position.

On the other hand, one cannot move from a $P$-position to another $P$-position using the moves in $\mathcal{V} \setminus \mathcal{M}$, so $\mathcal{V} \setminus \mathcal{M}$ satisfies (3). 

\[ \Box \]
The intuition behind Proposition 1 is that adjoining moves to a given game from \( P \)-positions to \( N \)-positions or vice versa, or from \( N \)-positions to \( N \)-positions, leaves the set of \( P \)-positions invariant, as long as no move from \( P \) to \( P \) is adjoined, and no cycles are formed. The conditions \( k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 1} \) in (2) prevent cycles. Note that the existence and uniqueness of \( V_{\max} \) is implied by Proposition 1.

From now on, we will analyze the structure of \( \mathcal{M} \).

An algorithm that determines whether a move \((x, y)\) is in \( \mathcal{M} \) was given in [3] for the original Wythoff \((\alpha = [1; 1, 1, 1, \ldots] = (1 + \sqrt{5})/2)\).

In this paper, we give a formula for all the \( y \)'s such that \((x, y) \in V_{\max}\) for a fixed \( x \), rather than only an algorithm that determines whether any specific element is in this set (as in [3]).

Observe that there are two ways to connect two \( P \)-positions, \((\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)\) and \((\lfloor \alpha m \rfloor, \lfloor \beta m \rfloor)\):

1. The direct way: \((\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor, \lfloor \beta n \rfloor - \lfloor \beta m \rfloor)\), possible when \( n > m \).
2. The crossed way: \((\lfloor \alpha n \rfloor - \lfloor \beta m \rfloor, \lfloor \beta n \rfloor - \lfloor \alpha m \rfloor)\), possible when \( \lfloor \alpha n \rfloor > \lfloor \beta m \rfloor \).

We define the set \( \mathcal{M}_1 \) as the set of moves that are obtained in the direct way, and we define \( \mathcal{M}_2 \) for the crossed way similarly. Notice that \( \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \).

We will analyze each of these sets separately.

Figure 1 shows a matrix \((a_{xy})\) where \( a_{xy} = 1 \) if \((x, y) \in \mathcal{M}_1\), \( a_{xy} = 2 \) if \((x, y) \in \mathcal{M}_2\), \( a_{xy} = 3 \) if \((x, y) \in \mathcal{M}_1 \cap \mathcal{M}_2\) and \( a_{xy} = 0 \) otherwise, for the case \( \alpha = [1; 1, 2, 3, \ldots] = 1.6977746\ldots, \beta = 2.4331274\ldots \).

## 2.1 Notation

For a set \( A \subseteq \mathbb{Z} \), let \( A-x = \{a-x : a \in A\} \) and \( A-x = (A-x) \cap \mathbb{Z}_{\geq 0} \).

Let \( x \in \mathbb{R} \). Denote its integer part by \( \lfloor x \rfloor \) and its fractional part by \( \{x\} \), so \( x = \lfloor x \rfloor + \{x\} \), \( \lfloor x \rfloor \in \mathbb{Z} \) and \( \{x\} \in [0, 1) \).

Every continued fraction alluded to in the sequel is a simple continued fraction (with numerators 1, denominators positive integers). See [7, ch. 10].

Let \( \Sigma \) be a finite alphabet of letters. Then, \( \Sigma^* \) is the free monoid over \( \Sigma \) and its elements are the finite words over \( \Sigma \). Let \( \varepsilon \in \Sigma^* \) denote the empty word. For \( w \in \Sigma^* \), let \( |w| \) denote the length of \( w \), counting multiplicities, and let \( |w|_\sigma \) denote the number of occurrences of the letter \( \sigma \in \Sigma \) in \( w \). We refer to the \( i \)-th letter of \( w \) by \( w(i) \) and we use the index 0 for the first letter.
In other words, $w = w(0)w(1) \cdots w(|w| - 1)$. General references about words and morphisms of words are [9], [10].

### 3 The set $\mathcal{M}_1$

Notice that $(x, y) \in \mathcal{M}_1$ if and only if $x = \lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor$ and $y = \lfloor \beta n \rfloor - \lfloor \beta m \rfloor$ for some $n > m$. Observe that $x = \lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor = \lfloor \alpha (n-m) \rfloor + a$, where $a = 1$ when $\{\alpha n\} < \{\alpha (n-m)\}$ and $a = 0$ otherwise. Similarly, we can write $y = \lfloor \beta (n-m) \rfloor + b$ where $b = 1$ if and only if $\{\beta n\} < \{\beta (n-m)\}$.

Let $\mathcal{X}(k)$ be the set of the pairs $(a, b)$ that are obtained by taking $n, m$ such that $n - m = k$. Then,

$$
\mathcal{M}_1 = \{(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b) : k \in \mathbb{Z}_{\geq 1}, \ (a, b) \in \mathcal{X}(k)\}.
$$

We now analyze the set $\mathcal{X}(k)$. For $n = k$ and $m = 0$, we get $(0, 0) \in \mathcal{X}(k)$ for every $k$. From now on, we assume $n > k$.

Let $\nu_0 = \{\alpha k\}, \xi_0 = \{\beta k\}$. Let $\mathbb{T}^2$ denote the torus $[0,1) \times [0,1)$, let $R_{ab} \subseteq \mathbb{T}^2$ be the rectangle defined in Table 1 and let $D = \{\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{>k}\}$. Then, $(a, b) \in \mathcal{X}(k)$ if and only if $R_{ab} \cap D \neq \emptyset$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<th>10</th>
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<th>12</th>
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</tr>
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<tr>
<td>$\lfloor \alpha n \rfloor$</td>
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<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td>$\lfloor \beta n \rfloor$</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>12</td>
<td>14</td>
<td>17</td>
<td>19</td>
<td>21</td>
<td>24</td>
<td>26</td>
<td>29</td>
<td>31</td>
</tr>
</tbody>
</table>

Figure 1: The sets $\mathcal{M}_1, \mathcal{M}_2$ for $\alpha = [1; 1, 2, 3, \ldots]$
We now consider two cases. The first case is when the only solution for the equation
\[ A\alpha + B\beta + C = 0, \quad A, B, C \in \mathbb{Z}, \]  
(4)
is \((A, B, C) = (0, 0, 0)\). In this case, Kronecker’s theorem (see, for example, [7, ch. 23]) guarantees that \( D \) is dense in \( T^2 \) and therefore \( X(k) = \{0, 1\} \times \{0, 1\} \).

We now turn to the second case. Note that (4) has a nontrivial solution if and only if \( \alpha \) is a root of a quadratic polynomial with integer coefficients, and this is true when the continued fraction of \( \alpha \) is periodic (see [7, ch. 10]).

Observe that if (4) has a nontrivial solution then there exist \( A, B, C \in \mathbb{Z} \) such that \( \gcd(A, B, C) = 1 \) and the solutions of (4) are \( \{(Az, Bz, Cz) : z \in \mathbb{Z}\} \). We call \((A, B, C)\) the primitive solution.

**Lemma 1.** Let \((A, B, C)\) be the primitive solution of (4) and let \( E := \{(\nu, \xi) \in T^2 : A\nu + B\xi \in \mathbb{Z}\} \). Then, the (topological) closure of \( D \) is \( E \).

**Proof.** Notice that \( A\{n\alpha\} + B\{n\beta\} = A(n\alpha - \lfloor n\alpha \rfloor) + B(n\beta - \lfloor n\beta \rfloor) = -nC - A\lfloor n\alpha \rfloor + B\lfloor n\beta \rfloor \in \mathbb{Z} \). Therefore, \( D \subseteq E \).

We prove the case \( \gcd(A, B) = 1 \). The case \( \gcd(A, B) > 1 \) follows easily from this case.

Take \( u, v \in \mathbb{Z} \) such that \( vA - uB = 1 \). Consider the continuous function \( f : E \to S^1 \) given by \((\nu, \xi) \mapsto \{uv + v\xi\} \) where \( S^1 \) is the circle \([0, 1)\). Then,
\[
M := \begin{pmatrix} A & B \\ u & v \end{pmatrix}, \quad |M| = \begin{vmatrix} A & B \\ u & v \end{vmatrix} = 1 \implies M^{-1} \in M_{2\times2}(\mathbb{Z}).
\]

This implies that \( f \) is a homeomorphism between \( E \) and \( S^1 \).

Let \( \gamma = u\alpha + v\beta \). The image of \( D \) under \( f \) is
\[
f[D] = \{un\alpha + vn\beta : n \in \mathbb{Z}_{>k}\} = \{\gamma n : n \in \mathbb{Z}_{>k}\}.
\]

If \( \gamma \in \mathbb{Q} \), then \( u\alpha + v\beta = c/d \) for some \( c, d \in \mathbb{Z} \). This implies that \((ud, v\beta, -c)\) is a solution for (4). Then \(|M| = 0\), which contradicts the fact

\[
\begin{array}{c|c|c}
(a, b) & R_{ab} & \\
\hline
(0, 1) & \{(\nu, \xi) \in T^2 : \nu > \nu_0, \xi < \xi_0\} & R_{10} \\
(1, 0) & \{(\nu, \xi) \in T^2 : \nu < \nu_0, \xi > \xi_0\} & R_{11} \\
(1, 1) & \{(\nu, \xi) \in T^2 : \nu < \nu_0, \xi < \xi_0\} & R_{00} \\
\end{array}
\]

Table 1: The rectangle \( R_{ab} \subseteq T^2 \)
that $|M| = 1$. Hence $\gamma \notin \mathbb{Q}$, and therefore $f[D]$ is dense in $S^1$ and $D$ is dense in $E$. \qed

**Example 1.** Figure 2 shows the set $E$ for three cases: (a) $2\alpha + 3\beta \in \mathbb{Z}$, (b) $2\alpha - 4\beta \in \mathbb{Z}$, (c) $\alpha - \beta \in \mathbb{Z}$. Notice that,

1. The direction of the lines depends on the sign of $AB$.
2. In (b), $\gcd(A, B) = 2$, and therefore $E$ is the union of two circles on the torus.

![Figure 2: Examples of the set $E$](image)

We can now complete the characterization of $\mathcal{X}(k)$: When $AB > 0$, since the slope is negative, we have $(0, 1), (1, 0) \in \mathcal{X}(k)$ for every $k$. We have $(1, 1) \in \mathcal{X}(k)$ only when $(\nu_0, \xi_0)$ is not on the leftmost segment (in other words, when $|A|\nu_0 > 1$ or $|B|\xi_0 > 1$). We can use similar arguments for the case $AB < 0$. The following table summarizes the results:

<table>
<thead>
<tr>
<th>Sign of $AB$</th>
<th>$(a, b)$</th>
<th>Condition for $(a, b) \in \mathcal{X}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB &gt; 0$</td>
<td>$(0, 1)$</td>
<td>Always</td>
</tr>
<tr>
<td></td>
<td>$(1, 0)$</td>
<td>Always</td>
</tr>
<tr>
<td></td>
<td>$(1, 1)$</td>
<td>$</td>
</tr>
<tr>
<td>$AB &lt; 0$</td>
<td>$(0, 1)$</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>$(1, 0)$</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>$(1, 1)$</td>
<td>Always</td>
</tr>
</tbody>
</table>

**Example 2.** Consider the case of generalized Wythoff: $\beta = \alpha + t$, $t \in \mathbb{Z}$. Then, $(1, -1, t)$ is the primitive solution (see Figure 2(c)). This fits into the case $AB < 0$ and since $|A| = |B| = 1$, $\mathcal{X}(k) = \{(0, 0), (1, 1)\}$ for every $k \in \mathbb{Z}_{\geq 1}$. We obtain $\mathcal{M}_1 = \{([\alpha k] + z, [\beta k] + z) : k \in \mathbb{Z}_{\geq 1}, z \in \{0, 1\}\}$. 

7
4 The set $\mathcal{M}_2$

4.1 The $\alpha$-word

It was shown in [3], that for the original Wythoff ($\alpha = [1; 1, 1, \ldots]$), there is a relation between the set $\mathcal{M}_2$ and the infinite Fibonacci word (the Fibonacci word is defined, for example, in [10, ch. 1]). We start by considering the natural generalization of the infinite Fibonacci word, $\mathcal{F}$, to any $\alpha$.

**Definition 1.** For $\alpha \in (1, \infty) \setminus \mathbb{Q}$, the $\alpha$-word, $W[\alpha]$, is the infinite word over $\{a, b\}$, for which the positions of the $a$’s are given by $\lfloor \alpha n \rfloor - 1$ ($n \in \mathbb{Z}_{\geq 1}$), and the positions of the $b$’s are given by $\lfloor \beta n \rfloor - 1$ ($n \in \mathbb{Z}_{\geq 1}$), where $1/\alpha + 1/\beta = 1$.

Notice that the two sequences: \{\lfloor \alpha n \rfloor - 1 : n \in \mathbb{Z}_{\geq 1}\}, \{\lfloor \beta n \rfloor - 1 : n \in \mathbb{Z}_{\geq 1}\}
are a pair of complementary Beatty sequences and therefore partition $\mathbb{Z}_{\geq 0}$, and so $W[\alpha]$ is well-defined.

**Example 3.**

$W[[1; 1, 1, 1, \ldots]] = ababaababaababaababaababaababaababaababaababaababaababa \cdots = \mathcal{F},$

$W[[1; 1, 2, 3, \ldots]] = ababaababaababaababaababaababaababaababaababaabababaa \cdots .$

We now give another definition that is based on morphisms of words:

**Definition 2.** Let $t \in \mathbb{Z}_{\geq 1}$. The morphism $\varphi_t : \{a, b\}^* \to \{a, b\}^*$ is defined by:

$\varphi_t(a) = a^t b, \quad \varphi_t(b) = a.$

**Definition 3.** Let $\tau_1, \tau_2, \ldots$ be an infinite sequence of morphisms such that for each $i$, $\tau_i(a)$ starts with an $a$. Define their infinite product $\tau_1 \tau_2 \cdots (a)$ to be the word:

$\lim_{n \to \infty} \tau_1 \tau_2 \cdots \tau_n(a).$

Note that since $\tau_1 \cdots \tau_n(a)$ is a prefix of $\tau_1 \cdots \tau_{n+1}(a)$, the limit in the previous definition is well-defined. If $\tau_i(\sigma) \neq \varepsilon$ and $|\tau_i(a)| > 1$ for every $i$ and $\sigma$, then $\tau_1 \tau_2 \cdots (a)$ is an infinite word.

**Theorem 1.** If $\alpha = [1; t_1, t_2, t_3, \ldots]$ then $W[\alpha] = \varphi_{t_1} \varphi_{t_2} \varphi_{t_3} \cdots (a)$.

To prove this theorem we will need the following lemma:
Lemma 2. Let \( \mu_1 \) be the morphism that sends \( a \mapsto b \) and \( b \mapsto a \) and let \( \mu_2 \) be the morphism that sends \( a \mapsto b'a \) and \( b \mapsto b \) for some \( t \in \mathbb{Z}_{\geq 1} \). Let \( \alpha \in (1, \infty) \setminus \mathbb{Q} \). Then,

\[
\mu_1(\mathcal{W}[^{\alpha}]) = \mathcal{W}[1 + 1/(\alpha - 1)], \quad \mu_2(\mathcal{W}[^{\alpha}]) = \mathcal{W}[^{\alpha + 1}].
\]

As a corollary,

\[
\varphi_t(\mathcal{W}[^{\alpha}]) = \mathcal{W}[1 + 1/(\alpha - 1 + t)].
\]

Proof. Let \( \beta = 1 + 1/(\alpha - 1) \) such that \( 1/\alpha + 1/\beta = 1 \). Therefore, the sequences \( \{\lfloor n\alpha \rfloor - 1\}_{n=1}^{\infty}, \{\lfloor n\beta \rfloor - 1\}_{n=1}^{\infty} \) partition the set \( \mathbb{Z}_{\geq 0} \). Since \( \{\lfloor n\alpha \rfloor - 1\}_{n=1}^{\infty} \) are the positions of the a’s of \( \mathcal{W}[^{\alpha}] \) and \( \{\lfloor n\beta \rfloor - 1\}_{n=1}^{\infty} \) are the positions of the a’s of \( \mu_1(\mathcal{W}[^{\alpha}]) \) and therefore \( \mu_1(\mathcal{W}[^{\alpha}]) = \mathcal{W}[\beta] \).

For \( \mu_2 \), notice that the positions of the a’s of \( \mathcal{W}[\alpha + t] \) are given by \( [(\alpha + t)n] - 1 = [\alpha n] - 1 + nt \). So in order to go from \( \mathcal{W}[^{\alpha}] \) to \( \mathcal{W}[\alpha + t] \) we have to insert \( b' \) to the left of each a. This is exactly the morphism \( \mu_2 \).

The corollary follows immediately:

\[
\varphi_t(\mathcal{W}[^{\alpha}]) = \mu_1 \mu_2(\mathcal{W}[^{\alpha}]) = \mu_1(\mathcal{W}[\alpha + t]) = \mathcal{W}[1 + 1/(\alpha - 1 + t)]. \quad \Box
\]

Proof of Theorem 1. Define \( \alpha_n = [1; t_{n+1}, t_{n+2}, \ldots] \) for \( n \in \mathbb{Z}_{\geq 0} \). The previous lemma implies that \( \varphi_{t_n}(\mathcal{W}[\alpha_n]) = \mathcal{W}[\alpha_{n-1}] \) and therefore

\[
\mathcal{W}[\alpha] = \mathcal{W}[\alpha_0] = \varphi_{t_1} \varphi_{t_2} \cdots \varphi_{t_n}(\mathcal{W}[\alpha_n]).
\]

Since \( a \) is a prefix of \( \mathcal{W}[\alpha_n] \), \( \varphi_{t_1} \varphi_{t_2} \cdots \varphi_{t_n}(a) \) is a prefix of \( \mathcal{W}[\alpha] \). Sending \( n \to \infty \), we get the requested result. \( \Box \)

Fix \( \alpha \in (1, 2) \setminus \mathbb{Q}, \alpha = [1; t_1, t_2, \ldots] \). Define a sequence of finite words: \( W_{-1} := b, W_0 := a \) and \( W_n := \varphi_{t_1} \cdots \varphi_{t_n}(a) \) for \( n \geq 1 \) and denote \( W := \mathcal{W}[\alpha] = \lim_{n \to \infty} W_n \). Let \( \alpha_n = [1; t_{n+1}, t_{n+2}, \ldots] \) as in the proof of Theorem 1.

For any word \( w \) of length \( \geq 2 \), write \( w = w^bw^e \) where \( |w^e| = 2 \).

The following proposition describes the basic properties of the sequence \( W_n \). These are the natural generalizations of known properties of the (finite) Fibonacci words.

**Proposition 2.**

(a) For \( n \geq 0 \), \( W_{n+1} = (W_n)^{t_{n+1}}W_{n-1} \).

(b) \( |W_n| = p_n, |W_n|_{a} = q_n \) where \( p_n/q_n \) are the convergents of the continued fraction of \( \alpha \).
(c). \( p_{-1} = 1, \ p_0 = 1, \ p_{n+1} = t_{n+1}p_n + p_{n-1} \) (for \( n \geq 0 \)).
(d). \( q_{-1} = 0, \ q_0 = 1, \ q_{n+1} = t_{n+1}q_n + q_{n-1} \) (for \( n \geq 0 \)).
(e). For \( n \geq -1 \), \( (W_nW_{n+1})^b = (W_{n+1}W_n)^b \).
(f). For \( n \geq 1 \), if \( 2 \mid n \), then \( (W_n)^e = ba \) and if \( 2 \nmid n \) then \( (W_n)^e = ab \).
(g). \( (W_n)^b \) is a palindrome for \( n \geq 1 \).

**Proof.** Items (a)-(d) follows from the definition of \( W_n \), and items (e)-(g) can be proven by induction on \( n \).

4.2 \( E_x \)

As we mentioned before, we want to find a formula for the elements of \( \mathcal{M}_2 \) in a fixed row, \( x \). Let \( E_x \) be the set of these positions: \( E_x = \{ y \geq x : (x, y) \in \mathcal{M}_2 \} \). Let \( g(n) = \lfloor \alpha n \rfloor \), \( h(n) = \lfloor \beta n \rfloor \). Notice that \( g^{-1}(n) = \lceil n/\alpha \rceil \) (when \( n \in \text{Im} \, g \)), \( h^{-1}(n) = \lceil n/\beta \rceil \) (when \( n \in \text{Im} \, h \)).

The following proposition describes the relation between the set \( E_x \) and the \( \alpha \)-word. Notice that [3] describes a simpler relation for the case \( \alpha = [1; 1, 1, \ldots] \). A similar relation can be given also for generalized Wythoff (\( \alpha = [1; t, t, \ldots] \), \( t \in \mathbb{Z}_{\geq 1} \). See Section 9.2), but unfortunately the case of an arbitrary \( \alpha \) is more complicated.

Let \( A_0^0 (B_0^0) \) be the set of positions of the \( a \)'s (\( b \)'s) of \( W \). The reason for this notation will become clear later. Then, \( B_0^0 \cap (A_0^0 \ominus x) \) is the set of \( s \)'s such that \( W(s) = b \) and \( W(s + x) = a \).

**Proposition 3.** Let \( x \in \mathbb{Z}_{\geq 1} \). Then,

\[
E_x = \{ hg^{-1}(s + x + 1) - gh^{-1}(s + 1) : s \in B_0^0 \cap (A_0^0 \ominus x) \}.
\]

**Proof.** Suppose that \( y \in E_x \). Then, \( y = h(n) - g(m) \) and \( x = g(n) - h(m) \).
Choose \( s = h(m) - 1 \). Then \( s \in B_0^0 \), \( s + x \in A_0^0 \), so \( s \in B_0^0 \cap (A_0^0 \ominus x) \).
Moreover, \( y = h(n) - g(m) = hg^{-1}g(n) - gh^{-1}h(m) = hg^{-1}(s + x + 1) - gh^{-1}(s + 1) \).

The other direction is similar. \( \square \)
5 The sets $A^m_i$, $B^m_i$

5.1 Motivation

As we saw in the last section, we have to analyze the set $B_0^0 \cap (A_0^0 - x)$. Consider the case $\alpha = [1; 1, 2, 3, \ldots]$, $x = 2$. We have $B_0^0 \cap (A_0^0 - 2) = \{3, 8, 13, 20, 25, 30, 37, \ldots\}$. In the following $\alpha$-word, these positions are shown as $B$: abaBaabaBaabaBaabaBaabaBaabaBaa \cdots. Theorem 1 implies that $W = \varphi_1 \varphi_2(W[\alpha])$, so $W$ consists of the blocks $\varphi_1 \varphi_2(a) = ababa$, $\varphi_1 \varphi_2(b) = ab$ and the order of the blocks is determined by $W[\alpha_2]$. Notice that the $B$’s above are exactly the second $b$’s of each block $ababa$. This fact will follow from the results of Section 7.

Therefore we would like to consider “higher resolutions” of the $\alpha$-word. These resolutions will be represented using the sets $A^m_i$, $B^m_i$. We will start by constructing some tools that will help us to define these sets.

5.2 Partitions and morphisms

Let $w$ be an infinite word over some finite alphabet $\Sigma$ such that all the letters of $\Sigma$ are in $w$. For every $\sigma \in \Sigma$, take the set $P_w(\sigma) := \{y \in \mathbb{Z}_{\geq 0} : w(y) = \sigma\}$. Observe that the sets $P_w(\sigma)$ for $\sigma \in \Sigma$ form a partition of $\mathbb{Z}_{\geq 0}$.

**Definition 4.** The partition induced by $w$ is $\mathcal{P}_w := \{P_w(\sigma) : \sigma \in \Sigma\}$.

**Remark.** In this paper we do not allow partitions that contain the empty set. Therefore, we defined $\mathcal{P}_w$ only when all the letters of $\Sigma$ appear in $w$.

**Definition 5.** Let $\Sigma$ be some finite alphabet and let $\tau : \Sigma^* \to \Sigma^*$ be a morphism. Consider the new alphabet $\Sigma_\tau := \{\sigma_i : \sigma \in \Sigma, 0 \leq i < |\tau(\sigma)|\}$. The indicator morphism of $\tau$ is the morphism $I_\tau : \Sigma^* \to \Sigma_\tau^*$ where $I_\tau(\sigma) = \sigma_0 \sigma_1 \cdots \sigma_{|\tau(\sigma)|-1}$ for every $\sigma \in \Sigma$.

**Example 4.** Consider the example in the “Motivation” section (Section 5.1). For $\tau = \varphi_1 \varphi_2$, we have $\Sigma_\tau = \{a_0, a_1, a_2, a_3, a_4, b_0, b_1\}$ and $a \mapsto I_\tau \rightarrow a_0 a_1 a_2 a_3 a_4$, $b \mapsto I_\tau \rightarrow b_0 b_1$. Observe that if $w = I_\tau(W[\alpha])$ then $P_w(a_3)$ is the set of the positions of the $B$’s, and therefore $P_w(a_3) = B_0^0 \cap (A_0^0 - 2)$.

Consider an infinite word $w$. The information in $I_\tau(w)$ is larger than the information in $\tau(w)$ in the sense that if we know the letter of $I_\tau(w)$ in some position, then we also know the letter of $\tau(w)$ in the same position. This is
stated formally in the following definition and proposition, using the notion of the induced partition.

**Definition 6.** Let \( \mathcal{A}, \mathcal{B} \) be two partitions of a set \( C \). We say that \( \mathcal{A} \) is **finer than** \( \mathcal{B} \), and we write \( \mathcal{A} \leq \mathcal{B} \), if for every set \( A \in \mathcal{A} \), there exists a set \( B \in \mathcal{B} \) such that \( A \subseteq B \).

It is easy to see that the relation “finer than” is a partial order relation over the set of partitions of \( C \).

**Proposition 4.** Let \( w \) be an infinite word and let \( \tau : \Sigma^* \to \Sigma^* \) be a morphism. Then \( \mathcal{P}_{I_\tau(w)} \leq \mathcal{P}_{\tau(w)} \).

**Proof.** This follows from the fact that \( \tau(w) \) and \( I_\tau(w) \) consist of blocks of the same lengths, in the same order, and in \( I_\tau \) each letter appears once. \( \square \)

### 5.3 Definition of \( \mathcal{A}_i^m, \mathcal{B}_i^m \)

Fix \( m \in \mathbb{Z}_{\geq 0} \). The morphism \( \Phi_m := \varphi_t \varphi_t \cdots \varphi_t \) satisfies: \( |\Phi_m(a)| = |W_m| = p_m \), \( |\Phi_m(b)| = |W_{m-1}| = p_{m-1} \) (see Proposition 2(b)). Therefore, the indicator morphism of \( \Phi_m \), \( \eta_m := I_{\Phi_m} \), maps: \( a \mapsto a_0 a_1 \cdots a_{p_{m-1}} \) and \( b \mapsto b_0 b_1 \cdots b_{p_{m-1}-1} \).

Let \( \mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m]) \) and denote the elements of the partition induced by \( \mathcal{H}_m \) by: \( \mathcal{A}_0^m, \mathcal{A}_1^m, \ldots, \mathcal{A}_{p_{m-1}}^m, \mathcal{B}_0^m, \mathcal{B}_1^m, \ldots, \mathcal{B}_{p_{m-1}-1}^m \) respectively.

**Example 5.** Consider Example 4 again. We have \( \tau = \Phi_2, I_\tau = \eta_2, w = \mathcal{H}_2 \) and \( \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 - 2) = P_w(a_3) = \mathcal{A}_3^2 \).

Observe that \( \mathcal{A}_0^0 (\mathcal{B}_0^0) \) is indeed the set of positions of the \( a \)'s (\( b \)'s) of \( \mathcal{W} \) as we defined before.

There is an equivalent construction for these sets, that uses a generalization of Zeckendorf sums, but we will not use it here. See Section 10.1.1 for details.

### 5.4 Properties

The following proposition gives a formula for the sets \( \mathcal{A}_i^m \):

**Proposition 5.** For \( m \in \mathbb{Z}_{\geq 0} \) and \( 0 \leq i < p_m \), we have:

\[
\mathcal{A}_i^m = \left\{ n \alpha_m n | p_{m-1} + n (p_m - p_{m-1}) - p_m + i : n \in \mathbb{Z}_{\geq 1} \right\}.
\]
Proof. Observe that the $n$-th $a_i$ of $\mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m])$ is generated by the $n$-th $a$ of $\mathcal{W}[\alpha_m]$. The position of this $a$ is $[\alpha_m n] − 1$. The first $[\alpha_m n] − 1$ letters of $\mathcal{W}[\alpha_m]$ contain $(n−1)$ a’s and $(\lfloor \alpha_m n \rfloor − n)$ b’s. Each $a$ generates $p_m$ letters, and each b generates $p_{m−1}$ letters. The claim follows. \hfill \Box

Observation 1. Let $m \in \mathbb{Z}_{≥0}$, $0 ≤ j ≤ i < p_m$. Then, $\mathcal{A}_i^m − j = \mathcal{A}_i^m \div j = \mathcal{A}_{i−j}^m$.

Proposition 6. $\mathcal{P}_{\mathcal{H}_0} ≥ \mathcal{P}_{\mathcal{H}_1} ≥ \mathcal{P}_{\mathcal{H}_2} ≥ \cdots$.

Proof. Fix $m \in \mathbb{Z}_{≥0}$. We have to show that $\mathcal{P}_{\mathcal{H}_m} ≥ \mathcal{P}_{\mathcal{H}_{m+1}}$.

Let $\tau = \varphi_{t_{m+1}}$. Notice that $|\Phi_m(w)| = |\eta_m(w)|$ for any word $w ∈ \{a, b\}^*$. In particular, $|\Phi_m(\sigma)| = |\eta_m(\tau(\sigma))|$ for $\sigma ∈ \{a, b\}$. This implies that $I_{\eta_m} = I_{\Phi_m} = \eta_m$, and so $\mathcal{H}_{m+1} = I_{\eta_m}(\mathcal{W}[\alpha_{m+1}])$. Using Proposition 4, we obtain that $\mathcal{P}_{\mathcal{H}_{m+1}} = \mathcal{P}_{\eta_m}(\mathcal{W}[\alpha_{m+1}]) ≤ \mathcal{P}_{\eta_m}(\mathcal{W}[\alpha_m]) = \mathcal{P}_{\mathcal{H}_m}$. \hfill \Box

Observation 2. If $m > 0$ and $y ∈ \mathcal{A}_i^m$ or $y ∈ \mathcal{B}_i^m$, then $\mathcal{W}(y) = \mathcal{W}(i)$.

Proof. The first part follows directly from the fact that $\mathcal{P}_{\mathcal{H}_m} ≤ \mathcal{P}_{\mathcal{H}_0} = \{\mathcal{A}_0^0, \mathcal{B}_0^0\}$ and the fact that $y, i ∈ \mathcal{A}_i^m$. For the second part, notice that both $W_{m+1}^m W_{m-1}, W_{m-1}$ are prefixes of $\mathcal{W}$. Therefore, $\mathcal{W}(i) = \mathcal{W}(i + t_{m+1} p_m)$ and the claim follows since $i + t_{m+1} p_m ∈ \mathcal{B}_i^m$. \hfill \Box

6 Shifts in $\mathcal{W}$

As we saw in Section 4.2, we have to examine the set $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \div x)$. We start with a simpler task: examining the set $\mathcal{A}_0^0 \Delta (\mathcal{A}_0^0 \div x)$, where $\Delta$ denotes the symmetric difference. This is the set of $y$’s for which $\mathcal{W}(y) ≠ \mathcal{W}(y + x)$.

Notice that $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \div x) = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \Delta (\mathcal{A}_0^0 \div x))$.

Our goal is to represent $\mathcal{A}_i^m \Delta (\mathcal{A}_0^0 \div x)$ using the basic sets $\mathcal{A}_i^m$ (for these sets we already have an explicit formula – Proposition 5).

We start with $x = p_k$ for $k ∈ \mathbb{Z}_{≥0}$ and then we generalize to an arbitrary $x ∈ \mathbb{Z}_{≥1}$.

6.1 Shifts by $p_k$, $k ∈ \mathbb{Z}_{≥0}$

Lemma 3. Let $k ∈ \mathbb{Z}_{≥0}$. If $0 ≤ i < p_{k+1} − 2$, then $\mathcal{W}(i) = \mathcal{W}(i + p_k)$. On the other hand, if $p_{k+1} - 2 ≤ i < p_{k+1}$, then $\mathcal{W}(i) ≠ \mathcal{W}(i + p_k)$.

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Proof. Notice that $W_{k+1}W_k$ is a prefix of $\mathcal{W}$. By Proposition 2(e), $(W_kW_{k+1})^b$ is also a prefix. This implies the first part. The second part follows from Proposition 2(f).

The following proposition describes the set $A_0^0 \Delta (A_0^0 \cdot p_k)$. It follows from the previous lemma and the fact that $\mathcal{H}_{k+1}$ consists of the blocks $a_0a_1 \cdots a_{p_{k+1}-1}$, $b_0b_1 \cdots b_{p_k-1}$.

Proposition 7. For $k \in \mathbb{Z}_{\geq 0}$, $A_0^0 \Delta (A_0^0 \cdot p_k) = A_{p_{k+1}-1}^{k+1} \cup A_{p_{k+1}-2}^{k+1}$.

6.2 Arbitrary $x \in \mathbb{Z}_{\geq 1}$

To answer the question for an arbitrary $x$, we will use the following idea: A generalization of Zeckendorf sums (see [13], [5], [6]) can be used to represent $x$ as a sum of elements from the set $\Pi := \{p_0, p_1, p_2, \ldots\}$. Then, we use Proposition 7 for each of the summands.

Apply the following algorithm on $x$: While $x \neq 0$, find the largest $k$ such that $p_k \leq x$ and subtract $p_k$ from $x$. Formally, define two sequences:

$$x_0 := x,$$

$$k_i := \max\{k \in \mathbb{Z}_{\geq 0} : p_k \leq x_{i-1}\} \quad (i \geq 1),$$

$$x_i := x_{i-1} - p_{k_i} \quad (i \geq 1).$$

Notice that if $x_i = 0$ for some $i$, then the two sequences $k_j, x_j$ are not defined for $i \geq j$. Denote this $i$ by $n$. Observe that we get a representation of $x$ as a sum of elements from $\Pi$: $x = p_{k_1} + p_{k_2} + \cdots + p_{k_n}$.

Example 6. Consider the case $\alpha = [1; 1, 2, 3, \ldots]$, $\Pi = \{1, 2, 5, 17, 73, \ldots\}$, $x = 12 = 5 + 5 + 2$. Here the algorithm yields:

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>12</td>
<td>7</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$k_i$</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$p_{k_i}$</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Let $1 \leq i \leq n$. Denote $X_i := (A_0^0 \cdot x_{i-1}) \Delta (A_0^0 \cdot x_i)$ and observe that $A_0^0 \Delta (A_0^0 \cdot x) = X_1 \Delta X_2 \Delta \cdots \Delta X_n$. Proposition 7 implies that

$$X_i = (A_0^0 \Delta (A_0^0 \cdot p_{k_i})) \cdot x_i = (A_{p_{k_i}+1}^{k_i+1} \cup A_{p_{k_i+1}-2}^{k_i+1}) \cdot x_i.$$
For the case in the previous example, we get:

\[ W(3) \]

By induction on \( X \)

\[ \text{Proof.} \]

Proposition 2(g) implies that \( \text{Observation 3.} \)

If \( x \) and by the last observation (for \( z \)
\[ \text{the induction hypothesis,} \]
\[ \text{Observation 4.} \]

If \( x \) \[ \text{Continue with the notation of the previous section. We have:} \]
\[ \text{The fact that} \]
\[ \text{We now consider three cases: (1)} \]

\[ \text{We now consider three cases: (1) } W(x-1) = b, (2) W(x-2) = b \text{ and} \]
\[ \text{(3) } W(x-1) = W(x-2) = a. \]
Consider the first case: For $1 \leq i < n$ we have $x_i \geq 1$ and by Observation 4,
$$W(p_{k_i+1} - x_i - 2) = W(x - 1) = b.$$ Notice that $b = W(x - 1) = W(x_{n-1} - 1) = W(p_{k_n} - 1)$. This means that $2 \nmid k_n$ (see Proposition 2(f)). Therefore, $W(p_{k_n+1} - x_n - 2) = W(p_{k_n+1} - 2) = b$.

Hence, for $1 \leq i \leq n$, $W(p_{k_i+1} - x_i - 2) = b$. Since $W$ does not contain $bb$ as a factor, we get that $W(p_{k_i+1} - x_i - 1) = a$. This implies
$$B_0^0 \cap (A_0^0 \setminus x) = \Delta_{i=1}^n A_{p_{k_i+1} - x_i - 2}^{k_i+1}.$$ The other cases are analyzed similarly. Formulas for the $x$’s of each case can be obtained by considering the blocks of $\mathcal{H}_1$. The following table summarizes the three cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>$W(x - 2), W(x - 1)$</th>
<th>$x - 2 \in \cdots$</th>
<th>$B_0^0 \cap (A_0^0 \setminus x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a, b$</td>
<td>$A_{t_1 - 1}^1$</td>
<td>$\Delta_{i=1}^n A_{p_{k_i+1} - x_i - 2}^{k_i+1}$</td>
</tr>
<tr>
<td>2</td>
<td>$b, a$</td>
<td>$A_{t_1}^1$</td>
<td>$\Delta_{i=1}^n A_{p_{k_i+1} - x_i - 1}^{k_i+1}$</td>
</tr>
<tr>
<td>3</td>
<td>$a, a$</td>
<td>$A_{t_1}^1$ ($i &lt; t_1 - 1$), $B_0^0 = A_{t_1+1}^2 (t_2)$</td>
<td>$A_{t_1}^1 = B_0^0$</td>
</tr>
</tbody>
</table>

**Example 8.** For the case described in Example 7, we have $W(12 - 1) = b$ and therefore this is Case 1. This implies $B_0^0 \cap (A_0^0 \setminus 12) = A_3^0 \Delta A_{t_3}^2 \Delta A_3^0$.

### 8 $B_0^0 \cap (A_0^0 \setminus x)$ as a disjoint union of basic sets

Our goal now is to represent $B_0^0 \cap (A_0^0 \setminus x)$ as a disjoint union of sets of the form $A_i^m$, instead of taking their symmetric difference as we did in Section 7. Such a representation seems to be much better. However, in order to attain this, we will have to understand better the structure formed by the sets $A_i^m$, $B_i^m$.

#### 8.1 The structure of $A_i^m$, $B_i^m$

Notice that $\mathcal{H}_m = \eta_m(W[\alpha_m]) = \eta_m \varphi_{t_m+1}(W[\alpha_{m+1}])$, so both $\mathcal{H}_m$, $\mathcal{H}_{m+1}$ consist of blocks of lengths $p_{m+1}, p_m$ in an order determined by $W[\alpha_{m+1}]$. 

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By considering these blocks we obtain:
\[
A_i^m = A_{i+1}^{m+1} \cup \cdots \cup A_{i+(t_m+1-1)p_m}^{m+1} \cup B_{i+1}^{m+1}, \quad B_i^m = A_{i+t_m+1p_m}^{m+1}.
\]

Therefore,
\[
A_i^m = A_{i+1}^{m+1} \cup \cdots \cup A_{i+(t_m+1-1)p_m}^{m+1} \cup A_{i+t_m+2p_{m+1}}^{m+2}.
\]  

(5)

**Definition 7.** A *partition tree* of a set $C \neq \emptyset$ is a tree, in which every node is a subset of $C$, the root is $C$, and for every node $A$, which is not a leaf, the set of children of $A$ form a partition of $A$.

Consider the tree of all the sets $A_i^m \subseteq B_i^0$, where there is an edge from $A_i^m$ to each of the sets in the right-hand side of (5). We denote this tree by $\mathcal{T}_\alpha$. Notice that the root of the tree is $A_{t_1}^1 = B_i^0$. Let $\text{pr} A$ denote the parent of a set $A$ in the tree. If $A$ is the root, we define $\text{pr} A := A$. Notice that $\mathcal{T}_\alpha$ is a partition tree.

**Example 9.** Figure 3 shows the tree $\mathcal{T}_\alpha$ for $\alpha = [1; 1, 2, 3, \ldots]$. For example, $\text{pr} A_{16}^3 = A_1^3$ and $\text{pr} A_3^4 = A_2^3$.

![Figure 3: $\mathcal{T}_\alpha$ for $\alpha = [1; 1, 2, 3, \ldots]$](image)

**Corollary 1.** Consider the node $A_i^m$ in $\mathcal{T}_\alpha$, where $A_i^m$ is not the root. We have
\[
\text{pr} A_i^m = A_{i \mod p_{m-1}}^m, \quad \text{where} \quad \overline{m} = \begin{cases} m-1, & i < p_{m-1} \cdot t_m \\ m-2, & i \geq p_{m-1} \cdot t_m \end{cases}.
\]

**Proof.** This follows directly from (5). \qed
8.2 The Chain Proposition

Notice that for Case 3 (see table on page 16) we have $B_0^0 \cap (A_0^0 - x) = A_{t_1}^1$. So we focus on the first two cases. Let $Z = 2$ for Case 1, and $Z = 1$ for Case 2. Denote $r_i := p_{h+1} - x_i - Z$. Then, $B_0^0 \cap (A_0^0 - x) = \Delta_{i=1}^n A_{r_i+1}^k$.

**Proposition 8.** For $1 \leq i < n$, $pr A_{r_i+1}^{k+1} \subseteq pr A_{r_i+1}^{k+1}$.

In order to prove Proposition 8 we first prove the following two lemmas:

**Lemma 4.** Let $1 \leq k \leq m$, $m \equiv k \pmod{2}$, $1 \leq i \leq p_k$. Then, $A_{p_m-i}^m \subseteq A_{p_k-i}^k$.

**Proof.** By Equation (5), we have that $A_{p_k-i}^k \supseteq A_{p_k-i+2}^{k+2}$ and $A_{p_k-i+2}^{k+2} \supseteq A_{p_k-i+4}^{k+4}$ and we get the following sequence:

$$A_{p_k-i}^k \supseteq A_{p_k-i+2}^{k+2} \supseteq A_{p_k-i+4}^{k+4} \supseteq \ldots.$$ 

Clearly $A_{p_m-i}^m$ is one of the elements of this sequence and so $A_{p_m-i}^m \subseteq A_{p_k-i}^k$.

**Lemma 5.** Let $k \geq 2$, $0 \leq i < p_k - p_{k-1}$. If both $A_i^k$, $A_{i+p_{k-1}}^k$ are nodes of $\mathcal{A}$, then $pr A_i^k \subseteq pr A_{i+p_{k-1}}^k$.

**Proof.** Corollary 1 implies that $pr A_i^k = A_j^{k_1}$, $pr A_{i+p_{k-1}}^k = A_j^{k_2}$ for some $j$, where $k_1, k_2 \in \{k - 1, k - 2\}$.

If $k_1 = k_2$, the claim holds. Otherwise, $k_1 = k - 1$, $k_2 = k - 2$. This implies $j < p_{k-2}$, and so $pr A_i^k = A_j^{k_1} \subseteq pr A_{j-1}^{k_1} = A_j^{k_2} = pr A_{i+p_{k-1}}^k$.

**Proof of Proposition 8.** We use the following notation:

$$a := x_i + Z, \quad k := k_i + 1,$$

$$b := x_{i+1} + Z, \quad \ell := k_{i+1} + 1.$$ 

In this notation, we have to show: $pr A_{p_k-a}^k \subseteq pr A_{p_{k-1}-b}^\ell$.

We have $p_{\ell-1} < a \leq p_{\ell} + 1$ and $p_{\ell} - b = p_{\ell} + p_{\ell-1} - a$. Note that all the sets that are mentioned in the proof are subsets of $B_0^0$ and therefore they are nodes in $\mathcal{A}$.

Consider the following 4 cases: (a) $\ell = 1$, (b) $k \equiv \ell + 1 \pmod{2}$, (c) $a \leq p_{\ell}$ and $k \equiv \ell \pmod{2}$, (d) $a = p_{\ell} + 1$ and $k \equiv \ell \pmod{2}$.

(a) is trivial. We show here the proof of (c). (b), (d) are proven similarly using applications of Lemma 4, Lemma 5 and Corollary 1.
Suppose that $a \leq p_\ell$ and $k \equiv \ell \pmod{2}$. Lemma 4 implies that $\mathcal{A}^k_{p_k-a} \subseteq \mathcal{A}^\ell_{p_\ell-a}$. Therefore, $\pr \mathcal{A}^k_{p_k-a} \subseteq \pr \mathcal{A}^\ell_{p_\ell-a}$. Lemma 5 implies that
\[ \pr \mathcal{A}^k_{p_k-a} \subseteq \pr \mathcal{A}^\ell_{p_\ell-a} \subseteq \pr \mathcal{A}^\ell_{p_\ell+1+p_\ell-a} = \pr \mathcal{A}^\ell_{p_\ell-b}. \]

\[ \Box \]

8.3 A disjoint union

Proposition 8 implies that the sets that participate in the symmetric difference satisfy the following property:
\[ \pr \mathcal{A}^{k_1+1}_{r_1} \subseteq \pr \mathcal{A}^{k_2+1}_{r_2} \subseteq \pr \mathcal{A}^{k_3+1}_{r_3} \subseteq \cdots \subseteq \pr \mathcal{A}^{k_n+1}_{r_n}. \] (6)

**Theorem 2.** The set $\mathcal{B}_0 \cap (\mathcal{A}_{0}^x - x)$ can be written as a disjoint union of $O(\Sigma_{i=1}^{k_1+1} t_i)$ sets of the form $\mathcal{A}_{i}^{m}$.

Notice that if $t_i < T$ for all $i \in \mathbb{Z}_{\geq 1}$, then the number of sets is $O(T \log x)$.

**Proof.** Define a partition subtree to be a subtree which is also a partition tree. In other words, every node of the subtree which is not a leaf, should have the same set of children as the same node in the original partition tree.

Consider the minimal partition subtree of $\mathcal{T}_a$ that contains the node $\mathcal{A}^{k_1+1}_{r_1}$. Denote it by $T_x$. This tree consists of the nodes $\pr^i \mathcal{A}^{k_1+1}_{r_1}$ ($i \in \mathbb{Z}_{\geq 1}$) and their children. Notice that (6) guarantees that all the sets $\mathcal{A}^{k_1+1}_{r_i}$ are nodes in the tree. The tree has at most $k_1 + 1$ layers, so the number of nodes is at most $\sum_{i=1}^{k_1+1} (t_i + 1)$. It is easy to see that in every finite partition tree, each element of the algebra (of sets) generated by the nodes, is a disjoint union of leaves. \[ \Box \]

Notice that Theorem 2 can be used to write an algorithm that gets $x$ and outputs a list of sets $\mathcal{A}_i^m$ whose disjoint union is $\mathcal{B}_0 \cap (\mathcal{A}_{0}^x - x)$: Compute the tree $T_x$ and mark the sets $\mathcal{A}^{k_1+1}_{r_i}$ in it. Visit the nodes of the tree, starting from the root, and if an internal node is marked, replace its mark with its children. Then, output the marked leaves.

**Example 10.** Consider the sets that appear in Example 8. The minimal partition subtree that contains $\mathcal{A}_3^3$ is shown in Figure 4. We have $\mathcal{B}_0 \cap (\mathcal{A}_{0}^x - 12) = \mathcal{A}_3^3 \Delta \mathcal{A}_{13}^3 \Delta \mathcal{A}_3^2 = \mathcal{A}_3^3 \cup \mathcal{A}_{13}^3$. 

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9 \( E_x \) as a union of basic sets

We saw that \( \mathcal{B}_0 \cap (\mathcal{A}_0^0 - x) = \bigcup_{j=1}^{n'} \mathcal{A}_i^{m_j} \) for some \( n', i_1, m_1, \ldots, i_{n'}, m_{n'} \).

Proposition 3 implies that \( E_x = \bigcup_{j=1}^{n'} F[\mathcal{A}_i^{m_j}] \) where \( F(s) = hg^{-1}(s + x + 1) - gh^{-1}(s + 1) \). In this section we give a somewhat better representation of \( E_x \).

9.1 The general case

We start by computing \( h^{-1}(s + 1) \) for \( s \in \mathcal{A}_i^m \subseteq \mathcal{B}_0^0 \). Suppose that \( s \) is the \( n \)-th element of \( \mathcal{A}_i^m \). It is generated (when applying \( \Phi_m \)) by the \( n \)-th \( a \) of \( W[\alpha_m] \). Let \( j = h^{-1}(i + 1) \) be the number of \( b \)'s in the first \( i + 1 \) letters of \( \Phi_m(a) \). Since the \( n \)-th \( a \) of \( W[\alpha_m] \) is in position \( [\alpha_m n] - 1 \), there are \( (n - 1) \) \( a \)'s and \( ([\alpha_m n] - n) \) \( b \)'s before this \( a \). Each \( a \) contributes (when applying \( \Phi_m \)) \( (p_m - q_m) \) \( b \)'s and each \( b \) contributes \( (p_m - 1 - q_m - 1) \) \( b \)'s. This implies:

\[
    h^{-1}(s + 1) = (p_m - q_m) \cdot (n - 1) + (p_{m-1} - q_{m-1}) \cdot ([\alpha_m n] - n) + j.
\]

In other words, there are constants \( A, B, C \in \mathbb{Z} \) such that \( h^{-1}(s + 1) = A[\alpha_m n] + Bn + C \).

In order to compute \( g^{-1}(s+x+1) \) we will need the following generalization of a proposition that appears in [3] (it is proven there for the case \( \alpha = [1; 1, 1, \ldots] \):

**Proposition 9.** If \( bua \) is a factor of \( W \) where \( n = |u| \) then \( |u|_a = |w|_a \) and \( |u|_b = |w|_b \) where \( w \) is the prefix of \( W \) of length \( n \).

**Proof.** It suffices to prove that \( |u|_b = |w|_b \) as \( |u| = |w| \). Denote by \( j \) the index of the first \( b \) of the \( bua \) factor.

Let \( X = \{i\beta: i \in \mathbb{Z}\} \). Notice that \( (z + 1, z + 2) \cap X \neq \emptyset \) if and only if \( W(z) = b \). Let \( f: \mathbb{R} \rightarrow \mathbb{Z}, f(x) = |(x, x + n) \cap X| \). In other words, \( f(x) \) is
the number of points from $X$ in the interval $(x, x + n)$. It is easy to see that $f$ is periodic with period $\beta$ and that $f$ is increasing on the interval $[0, \beta)$.

Notice that $|u|_b = f(j + 2)$ and $|w|_b = f(1)$. Since we have an $u$ after the $u$ it implies that $f(j + 3) \leq f(j + 2)$. We also know that there is a $b$ before the $u$ and therefore there is $r \in \mathbb{Z}$ such that $j + 1 < \beta r < j + 2$. Hence

$$\beta r < j + 2 < \beta r + 1 < j + 3 < \beta (r + 1).$$

But $f$ is increasing in the interval $[\beta r, \beta (r + 1))$ and so

$$f(j + 2) \leq f(\beta r + 1) \leq f(j + 3) \leq f(j + 2).$$

We conclude that $|w|_b = f(1) = f(\beta r + 1) = f(j + 2) = |u|_b$. \qed

Notice that $W(s - 1) = a$. We can give a formula for $g^{-1}(s)$ in a similar way to what we did for $h^{-1}(s + 1)$. Let $w$ be the prefix of length $x - 1$. By the last proposition, we have $g^{-1}(s + x + 1) = g^{-1}(s) + |w|_a + 1$ and so we get a formula for $g^{-1}(s + x + 1)$ that has the form $A'[\alpha_m n] + B'n + C'$.

We conclude that the set $E_x$ can be written as a union of sets of the form

$$\{ h(A'[\alpha_m n] + B'n + C') - g(A[\alpha_m n] + Bn + C) : n \in \mathbb{Z}_{\geq 1} \},$$

where $A, B, C, A', B', C' \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$.

**Example 11.** For $\alpha = [1; 1, 2, 3, \ldots]$ we have $E_{12} = F[A_8^3] \cup F[A_7^4]$ and

$$F[A_8^3] = \{ h(3[\alpha_3 n] + 7n) - g(2[\alpha_3 n] + 5n - 5) : n \in \mathbb{Z}_{\geq 1} \},$$

$$F[A_7^4] = \{ h(10[\alpha_4 n] + 33n + 7) - g(7[\alpha_4 n] + 23n) : n \in \mathbb{Z}_{\geq 1} \},$$

$$\alpha_3 = [1; 4, 5, 6, \ldots] \approx 1.23845, \quad \alpha_4 = [1; 5, 6, 7, \ldots] \approx 1.19369.$$

### 9.2 The case $\alpha = [1; t, t, t, \ldots]$

In turns out that in the case $\alpha = [1; t, t, t, \ldots]$ there is a simpler relation between $E_x$ and $B_0^0 \cap (A_0^0 \div x)$:

**Proposition 10.** Let $x \in \mathbb{Z}_{\geq 1}$. There exists $C \in \mathbb{Z}$ such that $F(s) = ts + C$ for any $s \in B_0^0 \cap (A_0^0 \div x)$.

**Proof.** Let $s \in B_0^0 \cap (A_0^0 \div x)$. Notice that $\beta = \alpha + t$ and so $h(y) = g(y) + yt$. Therefore, $h(g^{-1}(s + x + 1)) = g^{-1}(s + x + 1)t + s + x + 1$ and $g(h^{-1}(s + 1)) = s + 1 - h^{-1}(s + 1)t$. We also have $h^{-1}(s + 1) + g^{-1}(s) = s + 1$. This implies, $F(s) = h g^{-1}(s + x + 1) - g h^{-1}(s + 1) = x + [g^{-1}(s + x + 1) - g^{-1}(s) + s + 1] t$. Proposition 9 implies that $g^{-1}(s + x + 1) - g^{-1}(s)$ does not depend on $s$ and this completes the proof. \qed
10 Conclusion

We saw that the maximal set of moves that defines a game with $P$-positions $\left(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor \right)$ is $V \setminus (\mathcal{M}_1 \cup \mathcal{M}_2)$. We represented this set by a matrix $(a_{xy})$ where $a_{xy}$ indicates whether $(x, y) \in \mathcal{M}_1$ and whether $(x, y) \in \mathcal{M}_2$.

We examined the structure of any fixed row, $x$, of this matrix. The set $\mathcal{M}_1$ may contribute at most 4 elements for each row. We gave a description of $\mathcal{M}_1$ that facilitates computing these elements. For the set $\mathcal{M}_2$, we defined $E_x = \{y \geq x : (x, y) \in \mathcal{M}_2\}$. We saw that $E_x$ is related to the $\alpha$-word in the following manner: $E_x = F[B_0^0 \cap (A_0^0 - x)]$ where $F(s) = hg^{-1}(s + x + 1) - gh^{-1}(s + 1)$.

The next step was to investigate the set $B_0^0 \cap (A_0^0 - x)$. In order to do it, we wrote $x$ as a sum of $p_i$'s. In the process, we obtained two sequences: $x = x_0 > x_1 > \ldots > x_n = 0$ and $k_1 \geq k_2 \geq \ldots \geq k_n$, such that $\sum_{j=i+1}^n p_{k_j} = x_i$. It turned out that there are 3 cases:

1. When $W(x - 1) = b$, we have $B_0^0 \cap (A_0^0 - x) = \Delta_{i=1}^n A_{p_{k_i+1} - x_i - 2}^{k_i+1}$.
2. When $W(x - 2) = b$, we have $B_0^0 \cap (A_0^0 - x) = \Delta_{i=1}^n A_{p_{k_i+1} - x_i - 1}^{k_i+1}$.
3. When $W(x - 1) = W(x - 2) = a$, we have $B_0^0 \cap (A_0^0 - x) = B_0^0 = A_{t_1}^1$.

For the first two cases, we provided an algorithm that converts the symmetric difference to a disjoint union of sets of the form $A_i^m$.

Then we showed a way to simplify $F[A_i^m]$, and we concluded that $E_x$ is the union of sets of the form

$$\{ h(A'[\alpha_m n] + B'n + C') - g(A[\alpha_m n] + Bn + C) : n \in \mathbb{Z}_{\geq 1} \}.$$

Examples 6, 7, 8, 10, 11 show the process for the case $\alpha = [1; 1, 2, 3, \ldots]$ and $x = 12$.

10.1 Further directions of research

10.1.1 Zeckendorf sums

Let $x \in \mathbb{Z}_{\geq 0}$. It is well known (see, for example, [6] and [5]) that $x$ can be written as $x = \sum_{i=0}^\infty \tilde{x}_i p_i$ where $0 \leq \tilde{x}_i \leq t_{i+1}$ such that if $\tilde{x}_i = t_{i+1}$ for some $i > 0$ then $\tilde{x}_{i-1} = 0$. Moreover, this representation is unique.

**Definition 8.** For $x \in \mathbb{Z}_{\geq 0}$, define $R_m(x) = \sum_{i=0}^{m-1} \tilde{x}_i p_i$. 

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The following proposition, which we do not prove here, gives another definition for the sets $A_i^m$, $B_i^m$:

**Proposition 11.** $A_i^m = \{ x \in \mathbb{Z}_{\geq 0} : R_m(x) = i \text{ and } \bar{x}_m < t_{m+1} \}$ and $B_i^m = \{ x \in \mathbb{Z}_{\geq 0} : R_m(x) = i \text{ and } \bar{x}_m = t_{m+1} \}$.

This definition gives us another way to look at these sets. It is possible that one can rewrite the claims we proved here using the $\alpha$-word, and use the definition in Proposition 11 instead.

10.1.2 Finding a “nice” set of moves

For generalized Wythoff, we have a “nice” set of moves that defines the game: $\{(0, k) : k \in \mathbb{Z}_{\geq 1}\} \cup \{(k, \ell) : k, \ell \in \mathbb{Z}_{\geq 1}, 0 \leq \ell - k < t\}$. For $\alpha = [1; 1, t, 1, t, \ldots]$ there is also a “nice” set of moves (see [4]). However, for an arbitrary irrational $1 < \alpha < 2$, this is not the case. [8] shows the construction of such a set and here we described the maximal set, but neither can be considered “nice”. The question is whether such a “nice” set of moves exists for the case of an arbitrary $\alpha$ or for some subset of the possible $\alpha$’s.

**References**


