Extensions of Wythoff's Game

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Abstract

We determine the maximal set of moves for 2-pile take-away games with prescribed P-positions $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$ for $n \in \mathbb{Z}_{\geq 1}$ where $\alpha \in (1, 2)$ is irrational, $1/\alpha + 1/\beta = 1$. This was done in [3] for the special case α = golden ratio. We generalize the infinite Fibonacci word to an infinite word $\mathcal{W}$ with alphabet $\Sigma = \{a, b\}$, in which α replaces the golden ratio, and we analyze the set $\{s \in \mathbb{Z}_{\geq 0} : \mathcal{W}(s) = b,\ \mathcal{W}(s + x) = a\}$ for any fixed value of x.

1 Introduction

Generalized Wythoff (see [5]) is a two-player game, played on two piles of tokens. The two possible types of moves are: a. remove a positive amount of tokens from one pile, b. remove $k > 0$ tokens from one pile and $\ell > 0$ from the other, provided that $|k - \ell| < t$, where $t \in \mathbb{Z}_{\geq 1}$ is a parameter of the game. The player making the last move wins.

The case $t = 1$, in which the second type of move is to remove the same amount of tokens from both piles, is the classical Wythoff game [11], a modification of the game Nim. From among the extensive literature on Wythoff's game we mention just three: [2], [5], [12].

Since the game is finite, every position of the game is either an N-position – a position from which the **N**ext player can win, or a P-position – a position from which the **P**revious player can win. The game positions are encoded in the form (x, y), where x, y are the sizes of the piles and $x \leq y$. It was shown in [5] that the set of P-position, $\mathcal{P}$, for generalized Wythoff is $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, where $\alpha = [1; t, t, t, \ldots] = (2 - t + \sqrt{t^2 + 4})/2$ and

$\beta = 1 + 1/(\alpha - 1)$. Notice that the condition $\beta = 1 + 1/(\alpha - 1)$ is equivalent to $1/\alpha + 1/\beta = 1$; and when $\alpha = [1; t, t, t, \ldots]$, then $\beta = \alpha + t$.

We consider two games to be identical if they have the same set of P-positions. Let

$$\alpha^{-1} + \beta^{-1} = 1, \ \alpha \text{ irrational}, \ 0 < \alpha < \beta. \tag{1}$$

Then $1 < \alpha < 2 < \beta$. In this paper we seek the largest set of moves in games whose P-positions are $\{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)\}_{n \geq 0}$. The existence of such a game for an arbitrary irrational α was proven in [8].

For example, [4] describes a nice set of moves for $\alpha = [1; 1, t, 1, t, \ldots] = 1 + (\sqrt{t^2 + 4t} - t)/2$: A player can (a) remove a positive amount of tokens from one pile or (b) remove the same amount of tokens, k, from both piles as long as $k \notin \{2, 4, \ldots, 2t - 2\}$ or (c) remove $2t + 1$ tokens from one pile and $2t + 2$ tokens from the other.

It turns out that the largest set of moves is $\mathbb{V} \setminus \mathscr{M}$ where $\mathbb{V}$ is the set of all moves consisting of either taking $x > 0$ from a single pile, or else taking $x > 0$, $y > 0$ from both; and $\mathscr{M}$ is the set of moves that allow the players to move from one P-position to another.

We will consider the set of y's such that $(x, y) \in \mathscr{M}$ for any fixed x. It turns out that there is a strong relation between this set and a generalized version of the Fibonacci word, $\mathcal{W}$. In fact, we will have to investigate the set of y's such that $\mathcal{W}(y) = b$ and $\mathcal{W}(y + x) = a$.

This analysis can be done using a generalization of the Fibonacci numeration system (for information on numeration systems, see [6]), and also using techniques from the theory of words and morphisms of words. In this paper we chose the latter approach.

2 Preliminaries

An *invariant* game is a game for which the moves are playable from any position (see [4]). A *symmetric invariant* game is a game where the piles are unordered.

We consider symmetric invariant take-away games, played on two piles of tokens. We denote a position of the game by a pair (a, b) such that $a \leq b$. A move is also denoted by a pair (x, y) such that $x \leq y$. Notice that there can be two ways of playing this move from the position (a, b): to $(a - x, b - y)$ or to $(a - y, b - x)$ (we may need to change the order if $a - x > b - y$).

We assume throughout, without stating so explicitly, that we can never take away from any pile more than the pile size.

The set of moves $\mathbb{V}$ defined in the introduction can be written as $\mathbb{V} = \{(x, y) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : x \leq y,\ y \neq 0\}$. For any subset of moves $\mathscr{V} \subseteq \mathbb{V}$, let $\mathcal{P}(\mathscr{V})$ denote the set of P-positions of the game defined by $\mathscr{V}$ (the P- and N-positions of a game are defined in the introduction).

For example, for Generalized Wythoff,

$$\mathscr{V} = \{(0, k) : k \in \mathbb{Z}_{\geq 1}\} \cup \{(k, \ell) : k, l \in \mathbb{Z}_{\geq 1},\ 0 \leq \ell - k < t\}, \tag{2}$$

$$\mathcal{P}(\mathscr{V}) = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\},$$

where $\alpha = [1; t, t, t, \ldots]$ and $1/\alpha + 1/\beta = 1$.

Note that the definition of P- and N-positions implies that from a P-position the players can move only to N-positions and from an N-position there exists a move to a P-position. In particular, there is no move from any P-position to any other P-position. We say that the set $\mathcal{P}$ of P-positions of any given game constitute an *independent set*.

It was shown in [8], that for any irrational $\alpha \in (1, 2)$, there exists an invariant game with a set of moves, $\mathscr{V}$, such that $\mathcal{P}(\mathscr{V}) = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, where α, β satisfy (1). Notice that (1) implies that $\{\lfloor \alpha n \rfloor : n \in \mathbb{Z}_{\geq 1}\}$, $\{\lfloor \beta n \rfloor : n \in \mathbb{Z}_{\geq 1}\}$ are a pair of complementary Beatty sequences (see [1], [5]).

In this paper we study the following question: Fix an irrational $\alpha \in (1, 2)$. What is the maximal set of moves $\mathscr{V} \subseteq \mathbb{V}$ such that

$$\mathcal{P}(\mathscr{V}) = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}, \tag{3}$$

where $\beta = 1 + 1/(\alpha - 1)$?

Proposition 1. *Let $\mathscr{M} \subseteq \mathbb{V}$ be the subset of moves that allow the players to move from one P-position to another. The maximal set of moves, $\mathscr{V}_{max}$, that satisfies* (3) *is $\mathbb{V} \setminus \mathscr{M}$.*

Proof. Since $\mathcal{P}$ is an independent set, $\mathscr{M} \cap \mathscr{V} = \emptyset$ for every subset of moves $\mathscr{V}$ that satisfies (3). So $\mathscr{V} \subseteq \mathbb{V} \setminus \mathscr{M}$.

Take a set $\mathscr{V}_0$ that satisfies (3). The existence of an invariant game G with move set $\mathscr{V}_0$ satisfying (3) was proven in [8]. In particular, in G the move set $\mathscr{V}_0 \subseteq \mathbb{V} \setminus \mathscr{M}$ permits to move from every N-position into a P-position.

On the other hand, one cannot move from a P-position to another P-position using the moves in $\mathbb{V} \setminus \mathscr{M}$, so $\mathbb{V} \setminus \mathscr{M}$ satisfies (3). $\square$

The intuition behind Proposition 1 is that adjoining moves to a given game from P-positions to N-positions or vice versa, or from N-positions to N-positions, leaves the set of P-positions invariant, as long as no move from $\mathcal{P}$ to $\mathcal{P}$ is adjoined, and no cycles are formed. The conditions $k \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathbb{Z}_{\geq 1}$ in (2) prevent cycles. Note that the existence and uniqueness of $\mathscr{V}_{\max}$ is implied by Proposition 1.

From now on, we will analyze the structure of $\mathscr{M}$.

An algorithm that determines whether a move (x, y) is in $\mathscr{M}$ was given in [3] for the original Wythoff ($\alpha = [1; 1, 1, 1, \ldots] = (1 + \sqrt{5})/2$).

In this paper, we give a formula for all the y's such that $(x, y) \in \mathscr{V}_{\max}$ for a fixed x, rather than only an algorithm that determines whether any specific element is in this set (as in [3]).

Observe that there are two ways to connect two P-positions, $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$ and $(\lfloor \alpha m \rfloor, \lfloor \beta m \rfloor)$:

1. The direct way: $(\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor, \lfloor \beta n \rfloor - \lfloor \beta m \rfloor)$, possible when $n > m$.
2. The crossed way: $(\lfloor \alpha n \rfloor - \lfloor \beta m \rfloor, \lfloor \beta n \rfloor - \lfloor \alpha m \rfloor)$, possible when $\lfloor \alpha n \rfloor > \lfloor \beta m \rfloor$.

We define the set $\mathscr{M}_1$ as the set of moves that are obtained in the direct way, and we define $\mathscr{M}_2$ for the crossed way similarly. Notice that $\mathscr{M} = \mathscr{M}_1 \cup \mathscr{M}_2$. We will analyze each of these sets separately.

Figure 1 shows a matrix (a_{xy}) where $a_{xy} = 1$ if $(x, y) \in \mathscr{M}_1$, $a_{xy} = 2$ if $(x, y) \in \mathscr{M}_2$, $a_{xy} = 3$ if $(x, y) \in \mathscr{M}_1 \cap \mathscr{M}_2$ and $a_{xy} = 0$ otherwise, for the case $\alpha = [1; 1, 2, 3, \ldots] = 1.6977746...$, $\beta = 2.4331274....$

2.1 Notation

For a set $A \subseteq \mathbb{Z}$, let $A - x = \{a - x : a \in A\}$ and $A \dot{-} x = (A - x) \cap \mathbb{Z}_{\geq 0}$.

Let $x \in \mathbb{R}$. Denote its integer part by $\lfloor x \rfloor$ and its fractional part by $\{x\}$, so $x = \lfloor x \rfloor + \{x\}$, $\lfloor x \rfloor \in \mathbb{Z}$ and $\{x\} \in [0, 1)$.

Every continued fraction alluded to in the sequel is a *simple* continued fraction (with numerators 1, denominators positive integers). See [7, ch. 10].

Let Σ be a finite alphabet of letters. Then, Σ^* is the free monoid over Σ and its elements are the finite words over Σ. Let $\varepsilon \in \Sigma^*$ denote the empty word. For $w \in \Sigma^*$, let $|w|$ denote the length of w, counting multiplicities, and let $|w|_\sigma$ denote the number of occurrences of the letter $\sigma \in \Sigma$ in w. We refer to the i-th letter of w by $w(i)$ and we use the index 0 for the first letter.

$x \setminus y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	1	3	2	0	0	2	2	0	0	2	0	0	0	2	2	0	0	0	2	0	0	2	2	0	0
2		1	1	0	0	2	0	0	0	0	2	0	0	2	0	0	0	0	2	0	0	0	2	0	0	2
3			0	1	1	2	0	0	2	0	0	0	2	0	0	0	0	2	0	0	0	2	0	0	0	2
4				1	1	0	0	2	2	0	0	2	2	0	0	2	0	0	0	0	2	0	0	0	2	0
5					0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
6						0	1	1	1	1	2	0	0	2	2	0	0	2	2	0	0	0	2	0	0	0
7							0	0	1	1	0	0	0	2	0	0	0	2	0	0	0	0	0	0	0	2
8								0	0	0	0	1	3	0	0	2	0	0	0	0	2	0	0	0	2	0
9									0	0	0	1	1	0	0	2	0	0	2	2	0	0	0	2	0	0
10										0	0	0	0	1	1	0	0	0	0	0	0	0	2	0	0	0
11											0	0	0	1	1	0	1	3	0	0	2	0	2	0	0	2
12												0	0	0	0	0	1	1	0	0	2	0	0	0	0	0

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lfloor \alpha n \rfloor$	1	3	5	6	8	10	11	13	15	16	18	20	22
$\lfloor \beta n \rfloor$	2	4	7	9	12	14	17	19	21	24	26	29	31

Figure 1: The sets $\mathscr{M}_1, \mathscr{M}_2$ for $\alpha = [1; 1, 2, 3, \ldots]$

In other words, $w = w(0)w(1)\cdots w(|w|-1)$. General references about words and morphisms of words are [9], [10].

3 The set $\mathscr{M}_1$

Notice that $(x, y) \in \mathscr{M}_1$ if and only if $x = \lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor$ and $y = \lfloor \beta n \rfloor - \lfloor \beta m \rfloor$ for some $n > m$. Observe that $x = \lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor = \lfloor \alpha(n-m) \rfloor + a$, where $a = 1$ when $\{\alpha n\} < \{\alpha(n-m)\}$ and $a = 0$ otherwise. Similarly, we can write $y = \lfloor \beta(n-m) \rfloor + b$ where $b = 1$ if and only if $\{\beta n\} < \{\beta(n-m)\}$.

Let $\mathscr{X}(k)$ be the set of the pairs (a, b) that are obtained by taking n, m such that $n - m = k$. Then,

$$\mathscr{M}_1 = \{(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b) : k \in \mathbb{Z}_{\geq 1},\ (a, b) \in \mathscr{X}(k)\}.$$

We now analyze the set $\mathscr{X}(k)$. For $n = k$ and $m = 0$, we get $(0, 0) \in \mathscr{X}(k)$ for every k. From now on, we assume $n > k$.

Let $\nu_0 = \{\alpha k\}, \xi_0 = \{\beta k\}$. Let $\mathbb{T}^2$ denote the torus $[0, 1) \times [0, 1)$, let $R_{ab} \subseteq \mathbb{T}^2$ be the rectangle defined in Table 1 and let $D = \{(\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{>k}\}$. Then, $(a, b) \in \mathscr{X}(k)$ if and only if $R_{ab} \cap D \neq \emptyset$.

(a,b)	R_{ab}
$(0,1)$	$\{(\nu,\xi)\in\mathbb{T}^2 : \nu>\nu_0, \xi<\xi_0\}$
$(1,0)$	$\{(\nu,\xi)\in\mathbb{T}^2 : \nu<\nu_0, \xi>\xi_0\}$
$(1,1)$	$\{(\nu,\xi)\in\mathbb{T}^2 : \nu<\nu_0, \xi<\xi_0\}$

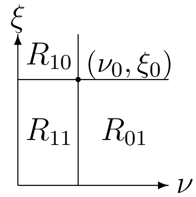


Table 1: The rectangle $R_{ab}\subseteq\mathbb{T}^2$

We now consider two cases. The first case is when the only solution for the equation

$$A\alpha+B\beta+C=0, \qquad A,B,C\in\mathbb{Z}, \tag{4}$$

is $(A,B,C)=(0,0,0)$. In this case, Kronecker's theorem (see, for example, [7, ch. 23]) guarantees that D is dense in $\mathbb{T}^2$ and therefore $\mathscr{X}(k)=\{0,1\}\times\{0,1\}$.

We now turn to the second case. Note that (4) has a nontrivial solution if and only if α is a root of a quadratic polynomial with integer coefficients, and this is true when the continued fraction of α is periodic (see [7, ch. 10]).

Observe that if (4) has a nontrivial solution then there exist $A,B,C\in\mathbb{Z}$ such that $\gcd(A,B,C)=1$ and the solutions of (4) are $\{(Az,Bz,Cz): z\in\mathbb{Z}\}$. We call (A,B,C) the *primitive solution*.

Lemma 1. *Let (A,B,C) be the primitive solution of (4) and let $E:=\{(\nu,\xi)\in\mathbb{T}^2 : A\nu+B\xi\in\mathbb{Z}\}$. Then, the (topological) closure of D is E.*

Proof. Notice that $A\{n\alpha\}+B\{n\beta\}=A(n\alpha-\lfloor n\alpha\rfloor)+B(n\beta-\lfloor n\beta\rfloor)=-nC-A\lfloor n\alpha\rfloor-B\lfloor n\beta\rfloor\in\mathbb{Z}$. Therefore, $D\subseteq E$.

We prove the case $\gcd(A,B)=1$. The case $\gcd(A,B)>1$ follows easily from this case.

Take $u,v\in\mathbb{Z}$ such that $vA-uB=1$. Consider the continuous function $f:E\to S^1$ given by $(\nu,\xi)\mapsto\{u\nu+v\xi\}$ where S^1 is the circle $[0,1)$. Then,

$$M:=\begin{pmatrix}A & B\\ u & v\end{pmatrix},\quad |M|=\begin{vmatrix}A & B\\ u & v\end{vmatrix}=1\implies M^{-1}\in M_{2\times 2}(\mathbb{Z}).$$

This implies that f is a homeomorphism between E and S^1.

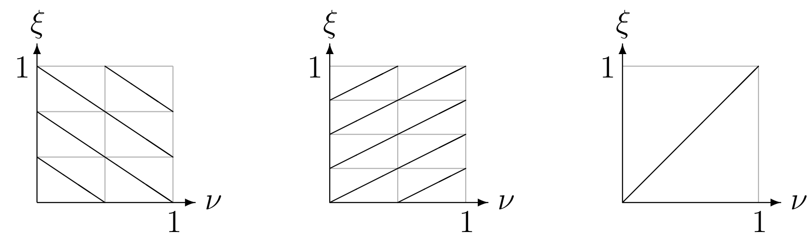
Let $\gamma=u\alpha+v\beta$. The image of D under f is

$$f[D]=\big\{\{un\alpha+vn\beta\}: n\in\mathbb{Z}_{>k}\big\}=\big\{\{\gamma n\}: n\in\mathbb{Z}_{>k}\big\}.$$

If $\gamma\in\mathbb{Q}$, then $u\alpha+v\beta=c/d$ for some $c,d\in\mathbb{Z}$. This implies that $(ud,vd,-c)$ is a solution for (4). Then $|M|=0$, which contradicts the fact that $|M| = 1$. Hence $\gamma \notin \mathbb{Q}$, and therefore $f[D]$ is dense in S^1 and D is dense in E. □

Example 1. Figure 2 shows the set E for three cases: (a) $2\alpha + 3\beta \in \mathbb{Z}$, (b) $2\alpha - 4\beta \in \mathbb{Z}$, (c) $\alpha - \beta \in \mathbb{Z}$. Notice that,

1. The direction of the lines depends on the sign of AB.
2. In (b), $\gcd(A, B) = 2$, and therefore E is the union of two circles on the torus.



$2\alpha + 3\beta \in \mathbb{Z}$ $AB > 0$ (a) $2\alpha - 4\beta \in \mathbb{Z}$ $AB < 0$ (b) $\alpha - \beta \in \mathbb{Z}$ $AB < 0$ (c)

Figure 2: Examples of the set E

We can now complete the characterization of $\mathscr{X}(k)$: When $AB > 0$, since the slope is negative, we have $(0, 1), (1, 0) \in \mathscr{X}(k)$ for every k. We have $(1, 1) \in \mathscr{X}(k)$ only when (ν_0, ξ_0) is not on the leftmost segment (in other words, when $|A|\nu_0 > 1$ or $|B|\xi_0 > 1$). We can use similar arguments for the case $AB < 0$. The following table summarizes the results:

Sign of AB	(a, b)	Condition for $(a, b) \in \mathscr{X}(k)$
$AB > 0$	$(0, 1)$	Always
	$(1, 0)$	Always
	$(1, 1)$	$|A|\nu_0 > 1$ or $|B|\xi_0 > 1$
$AB < 0$	$(0, 1)$	$|A|(1 - \nu_0) > 1$ or $|B|\xi_0 > 1$
	$(1, 0)$	$|A|\nu_0 > 1$ or $|B|(1 - \xi_0) > 1$
	$(1, 1)$	Always

Example 2. Consider the case of generalized Wythoff: $\beta = \alpha + t$, $t \in \mathbb{Z}$. Then, $(1, -1, t)$ is the primitive solution (see Figure 2(c)). This fits into the case $AB < 0$ and since $|A| = |B| = 1$, $\mathscr{X}(k) = \{(0, 0), (1, 1)\}$ for every $k \in \mathbb{Z}_{\geq 1}$. We obtain $\mathscr{M}_1 = \{(\lfloor \alpha k \rfloor + z, \lfloor \beta k \rfloor + z) : k \in \mathbb{Z}_{\geq 1}, z \in \{0, 1\}\}$.

4 The set $\mathscr{M}_2$

4.1 The α-word

It was shown in [3], that for the original Wythoff ($\alpha = [1; 1, 1, \ldots]$), there is a relation between the set $\mathscr{M}_2$ and the infinite Fibonacci word (the Fibonacci word is defined, for example, in [10, ch. 1]). We start by considering the natural generalization of the infinite Fibonacci word, $\mathcal{F}$, to any α.

Definition 1. For $\alpha \in (1, \infty) \setminus \mathbb{Q}$, the *$\alpha$-word*, $\mathcal{W}[\alpha]$, is the infinite word over $\{a, b\}$, for which the positions of the a's are given by $\lfloor \alpha n \rfloor - 1$ ($n \in \mathbb{Z}_{\geq 1}$), and the positions of the b's are given by $\lfloor \beta n \rfloor - 1$ ($n \in \mathbb{Z}_{\geq 1}$), where $1/\alpha + 1/\beta = 1$.

Notice that the two sequences: $\{\lfloor \alpha n \rfloor - 1 : n \in \mathbb{Z}_{\geq 1}\}$, $\{\lfloor \beta n \rfloor - 1 : n \in \mathbb{Z}_{\geq 1}\}$ are a pair of complementary Beatty sequences and therefore partition $\mathbb{Z}_{\geq 0}$, and so $\mathcal{W}[\alpha]$ is well-defined.

Example 3.

$$\begin{aligned}
\mathcal{W}[[1; 1, 1, 1, \ldots]] &= abaababaabaababaababaabaababaabaababaaba \cdots = \mathcal{F}, \\
\mathcal{W}[[1; 1, 2, 3, \ldots]] &= ababaababaababaabababaababaababaabababaa \cdots .
\end{aligned}$$

We now give another definition that is based on morphisms of words:

Definition 2. Let $t \in \mathbb{Z}_{\geq 1}$. The morphism $\varphi_t : \{a, b\}^* \to \{a, b\}^*$ is defined by:

$$\varphi_t(a) = a^t b, \qquad \varphi_t(b) = a.$$

Definition 3. Let $\tau_1, \tau_2, \ldots$ be an infinite sequence of morphisms such that for each i, $\tau_i(a)$ starts with an a. Define their *infinite product* $\tau_1 \tau_2 \cdots (a)$ to be the word:

$$\lim_{n \to \infty} \tau_1 \tau_2 \cdots \tau_n(a).$$

Note that since $\tau_1 \cdots \tau_n(a)$ is a prefix of $\tau_1 \cdots \tau_{n+1}(a)$, the limit in the previous definition is well-defined. If $\tau_i(\sigma) \neq \varepsilon$ and $|\tau_i(a)| > 1$ for every i and σ, then $\tau_1 \tau_2 \cdots (a)$ is an *infinite* word.

Theorem 1. *If* $\alpha = [1; t_1, t_2, t_3, \ldots]$ *then* $\mathcal{W}[\alpha] = \varphi_{t_1} \varphi_{t_2} \varphi_{t_3} \cdots (a)$.

To prove this theorem we will need the following lemma:

Lemma 2. *Let μ_1 be the morphism that sends $a \mapsto b$ and $b \mapsto a$ and let μ_2 be the morphism that sends $a \mapsto b^t a$ and $b \mapsto b$ for some $t \in \mathbb{Z}_{\geq 1}$. Let $\alpha \in (1, \infty) \setminus \mathbb{Q}$. Then,*

$$\mu_1(\mathcal{W}[\alpha]) = \mathcal{W}[1 + 1/(\alpha - 1)], \qquad \mu_2(\mathcal{W}[\alpha]) = \mathcal{W}[\alpha + t].$$

As a corollary,

$$\varphi_t(\mathcal{W}[\alpha]) = \mathcal{W}[1 + 1/(\alpha - 1 + t)].$$

Proof. Let $\beta = 1 + 1/(\alpha - 1)$ such that $1/\alpha + 1/\beta = 1$. Therefore, the sequences $\{\lfloor n\alpha \rfloor - 1\}_{n=1}^{\infty}$, $\{\lfloor n\beta \rfloor - 1\}_{n=1}^{\infty}$ partition the set $\mathbb{Z}_{\geq 0}$. Since $\{\lfloor n\alpha \rfloor - 1\}_{n=1}^{\infty}$ are the positions of the a's of $\mathcal{W}[\alpha]$ then $\{\lfloor n\beta \rfloor - 1\}_{n=1}^{\infty}$ are the positions of the a's of $\mu_1(\mathcal{W}[\alpha])$ and therefore $\mu_1(\mathcal{W}[\alpha]) = \mathcal{W}[\beta]$.

For μ_2, notice that the positions of the a's of $\mathcal{W}[\alpha + t]$ are given by $\lfloor (\alpha + t)n \rfloor - 1 = \lfloor \alpha n \rfloor - 1 + nt$. So in order to go from $\mathcal{W}[\alpha]$ to $\mathcal{W}[\alpha + t]$ we have to insert b^t to the left of each a. This is exactly the morphism μ_2.

The corollary follows immediately:

$$\varphi_t(\mathcal{W}[\alpha]) = \mu_1\mu_2(\mathcal{W}[\alpha]) = \mu_1(\mathcal{W}[\alpha + t]) = \mathcal{W}[1 + 1/(\alpha - 1 + t)]. \qquad \square$$

Proof of Theorem 1. Define $\alpha_n = [1; t_{n+1}, t_{n+2}, \ldots]$ for $n \in \mathbb{Z}_{\geq 0}$. The previous lemma implies that $\varphi_{t_n}(\mathcal{W}[\alpha_n]) = \mathcal{W}[\alpha_{n-1}]$ and therefore

$$\mathcal{W}[\alpha] = \mathcal{W}[\alpha_0] = \varphi_{t_1}\varphi_{t_2} \cdots \varphi_{t_n}(\mathcal{W}[\alpha_n]).$$

Since a is a prefix of $\mathcal{W}[\alpha_n]$, $\varphi_{t_1}\varphi_{t_2} \cdots \varphi_{t_n}(a)$ is a prefix of $\mathcal{W}[\alpha]$. Sending $n \to \infty$, we get the requested result. $\square$

Fix $\alpha \in (1, 2) \setminus \mathbb{Q}$, $\alpha = [1; t_1, t_2, \ldots]$. Define a sequence of finite words: $W_{-1} := b$, $W_0 := a$ and $W_n := \varphi_{t_1} \cdots \varphi_{t_n}(a)$ for $n \geq 1$ and denote $\mathcal{W} := \mathcal{W}[\alpha] = \lim_{n \to \infty} W_n$. Let $\alpha_n = [1; t_{n+1}, t_{n+2}, \ldots]$ as in the proof of Theorem 1.

For any word w of length ≥ 2, write $w = w^b w^e$ where $|w^e| = 2$.

The following proposition describes the basic properties of the sequence W_n. These are the natural generalizations of known properties of the (finite) Fibonacci words.

Proposition 2.

(a). *For $n \geq 0$, $W_{n+1} = (W_n)^{t_{n+1}} W_{n-1}$.*

(b). *$|W_n| = p_n, |W_n|_a = q_n$ where p_n/q_n are the convergents of the continued fraction of α.*

(c). $p_{-1} = 1$, $p_0 = 1$, $p_{n+1} = t_{n+1}p_n + p_{n-1}$ *(for $n \geq 0$).*

(d). $q_{-1} = 0$, $q_0 = 1$, $q_{n+1} = t_{n+1}q_n + q_{n-1}$ *(for $n \geq 0$).*

(e). *For $n \geq -1$, $(W_n W_{n+1})^b = (W_{n+1} W_n)^b$.*

(f). *For $n \geq 1$, if $2 \mid n$, then $(W_n)^e = ba$ and if $2 \nmid n$ then $(W_n)^e = ab$.*

(g). *$(W_n)^b$ is a palindrome for $n \geq 1$.*

Proof. Items (a)-(d) follows from the definition of W_n, and items (e)-(g) can be proven by induction on n. □

4.2 E_x

As we mentioned before, we want to find a formula for the elements of $\mathscr{M}_2$ in a fixed row, x. Let E_x be the set of these positions: $E_x = \{y \geq x : (x, y) \in \mathscr{M}_2\}$. Let $g(n) = \lfloor \alpha n \rfloor$, $h(n) = \lfloor \beta n \rfloor$. Notice that $g^{-1}(n) = \lceil n/\alpha \rceil$ (when $n \in \operatorname{Im} g$), $h^{-1}(n) = \lceil n/\beta \rceil$ (when $n \in \operatorname{Im} h$).

The following proposition describes the relation between the set E_x and the α-word. Notice that [3] describes a simpler relation for the case $\alpha = [1; 1, 1, \ldots]$. A similar relation can be given also for generalized Wythoff ($\alpha = [1; t, t, \ldots]$, $t \in \mathbb{Z}_{\geq 1}$. See Section 9.2), but unfortunately the case of an arbitrary α is more complicated.

Let $\mathcal{A}_0^0$ ($\mathcal{B}_0^0$) be the set of positions of the a's (b's) of $\mathcal{W}$. The reason for this notation will become clear later. Then, $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$ is the set of s's such that $\mathcal{W}(s) = b$ and $\mathcal{W}(s + x) = a$.

Proposition 3. *Let $x \in \mathbb{Z}_{\geq 1}$. Then,*

$$E_x = \{hg^{-1}(s + x + 1) - gh^{-1}(s + 1) : s \in \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)\}.$$

Proof. Suppose that $y \in E_x$. Then, $y = h(n) - g(m)$ and $x = g(n) - h(m)$. Choose $s = h(m) - 1$. Then $s \in \mathcal{B}_0^0$, $s + x \in \mathcal{A}_0^0$, so $s \in \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$. Moreover, $y = h(n) - g(m) = hg^{-1}g(n) - gh^{-1}h(m) = hg^{-1}(s + x + 1) - gh^{-1}(s + 1)$.

The other direction is similar. □

5 The sets $\mathcal{A}_i^m$, $\mathcal{B}_i^m$

5.1 Motivation

As we saw in the last section, we have to analyze the set $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$. Consider the case $\alpha = [1; 1, 2, 3, \ldots]$, $x = 2$. We have $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} 2) = \{3, 8, 13, 20, 25, 30, 37, \ldots\}$. In the following α-word, these positions are shown as $\mathbf{B}$: $aba\mathbf{B}aaba\mathbf{B}aaba\mathbf{B}aababa\mathbf{B}aaba\mathbf{B}aaba\mathbf{B}aababa\mathbf{B}aa\cdots$. Theorem 1 implies that $\mathcal{W} = \varphi_1\varphi_2(\mathcal{W}[\alpha_2])$, so $\mathcal{W}$ consists of the blocks $\varphi_1\varphi_2(a) = ababa$, $\varphi_1\varphi_2(b) = ab$ and the order of the blocks is determined by $\mathcal{W}[\alpha_2]$. Notice that the $\mathbf{B}$'s above are exactly the second b's of each block $ababa$. This fact will follow from the results of Section 7.

Therefore we would like to consider "higher resolutions" of the α-word. These resolutions will be represented using the sets $\mathcal{A}_i^m$, $\mathcal{B}_i^m$. We will start by constructing some tools that will help us to define these sets.

5.2 Partitions and morphisms

Let w be an infinite word over some finite alphabet Σ such that all the letters of Σ are in w. For every $\sigma \in \Sigma$, take the set $P_w(\sigma) := \{y \in \mathbb{Z}_{\geq 0} : w(y) = \sigma\}$. Observe that the sets $P_w(\sigma)$ for $\sigma \in \Sigma$ form a partition of $\mathbb{Z}_{\geq 0}$.

Definition 4. The *partition induced by w* is $\mathscr{P}_w := \{P_w(\sigma) : \sigma \in \Sigma\}$.

Remark. In this paper we do not allow partitions that contain the empty set. Therefore, we defined $\mathscr{P}_w$ only when all the letters of Σ appear in w.

Definition 5. Let Σ be some finite alphabet and let $\tau : \Sigma^* \to \Sigma^*$ be a morphism. Consider the new alphabet $\Sigma_\tau := \{\sigma_i : \sigma \in \Sigma,\ 0 \leq i < |\tau(\sigma)|\}$. The *indicator morphism* of τ is the morphism $I_\tau : \Sigma^* \to \Sigma_\tau^*$ where $I_\tau(\sigma) = \sigma_0\sigma_1 \cdots \sigma_{|\tau(\sigma)|-1}$ for every $\sigma \in \Sigma$.

Example 4. Consider the example in the "Motivation" section (Section 5.1). For $\tau = \varphi_1\varphi_2$, we have $\Sigma_\tau = \{a_0, a_1, a_2, a_3, a_4, b_0, b_1\}$ and $a \xmapsto{I_\tau} a_0a_1a_2a_3a_4$, $b \xmapsto{I_\tau} b_0b_1$. Observe that if $w = I_\tau(\mathcal{W}[\alpha_2])$ then $P_w(a_3)$ is the set of the positions of the $\mathbf{B}$'s, and therefore $P_w(a_3) = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} 2)$.

Consider an infinite word w. The information in $I_\tau(w)$ is larger than the information in $\tau(w)$ in the sense that if we know the letter of $I_\tau(w)$ in some position, then we also know the letter of $\tau(w)$ in the same position. This is stated formally in the following definition and proposition, using the notion of the induced partition.

Definition 6. Let $\mathscr{A}$, $\mathscr{B}$ be two partitions of a set C. We say that $\mathscr{A}$ is *finer than* $\mathscr{B}$, and we write $\mathscr{A} \leq \mathscr{B}$, if for every set $A \in \mathscr{A}$, there exists a set $B \in \mathscr{B}$ such that $A \subseteq B$.

It is easy to see that the relation "finer than" is a partial order relation over the set of partitions of C.

Proposition 4. *Let w be an infinite word and let $\tau : \Sigma^* \to \Sigma^*$ be a morphism. Then $\mathscr{P}_{I_\tau(w)} \leq \mathscr{P}_{\tau(w)}$.*

Proof. This follows from the fact that $\tau(w)$ and $I_\tau(w)$ consist of blocks of the same lengths, in the same order, and in I_τ each letter appears once. □

5.3 Definition of $\mathcal{A}_i^m$, $\mathcal{B}_i^m$

Fix $m \in \mathbb{Z}_{\geq 0}$. The morphism $\Phi_m := \varphi_{t_1}\varphi_{t_2}\cdots\varphi_{t_m}$ satisfies: $|\Phi_m(a)| = |W_m| = p_m$, $|\Phi_m(b)| = |W_{m-1}| = p_{m-1}$ (see Proposition 2(b)). Therefore, the indicator morphism of Φ_m, $\eta_m := I_{\Phi_m}$, maps: $a \overset{\eta_m}{\longmapsto} a_0a_1\cdots a_{p_m-1}$ and $b \overset{\eta_m}{\longmapsto} b_0b_1\cdots b_{p_{m-1}-1}$.

Let $\mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m])$ and denote the elements of the partition induced by $\mathcal{H}_m$ by: $\mathcal{A}_0^m, \mathcal{A}_1^m, \cdots, \mathcal{A}_{p_m-1}^m, \mathcal{B}_0^m, \mathcal{B}_1^m, \cdots, \mathcal{B}_{p_{m-1}-1}^m$ respectively.

Example 5. Consider Example 4 again. We have $\tau = \Phi_2$, $I_\tau = \eta_2$, $w = \mathcal{H}_2$ and $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} 2) = P_w(a_3) = \mathcal{A}_3^2$.

Observe that $\mathcal{A}_0^0$ ($\mathcal{B}_0^0$) is indeed the set of positions of the a's (b's) of $\mathcal{W}$ as we defined before.

There is an equivalent construction for these sets, that uses a generalization of Zeckendorf sums, but we will not use it here. See Section 10.1.1 for details.

5.4 Properties

The following proposition gives a formula for the sets $\mathcal{A}_i^m$:

Proposition 5. *For $m \in \mathbb{Z}_{\geq 0}$ and $0 \leq i < p_m$, we have:*

$$\mathcal{A}_i^m = \{\lfloor \alpha_m n \rfloor p_{m-1} + n(p_m - p_{m-1}) - p_m + i : n \in \mathbb{Z}_{\geq 1}\}.$$

Proof. Observe that the n-th a_i of $\mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m])$ is generated by the n-th a of $\mathcal{W}[\alpha_m]$. The position of this a is $\lfloor \alpha_m n \rfloor - 1$. The first $\lfloor \alpha_m n \rfloor - 1$ letters of $\mathcal{W}[\alpha_m]$ contain $(n-1)$ a's and $(\lfloor \alpha_m n \rfloor - n)$ b's. Each a generates p_m letters, and each b generates p_{m-1} letters. The claim follows. □

Observation 1. *Let $m \in \mathbb{Z}_{\geq 0}$, $0 \leq j \leq i < p_m$. Then, $\mathcal{A}_i^m - j = \mathcal{A}_i^m \dot{-} j = \mathcal{A}_{i-j}^m$.*

Proposition 6. $\mathscr{P}_{\mathcal{H}_0} \geq \mathscr{P}_{\mathcal{H}_1} \geq \mathscr{P}_{\mathcal{H}_2} \geq \cdots$.

Proof. Fix $m \in \mathbb{Z}_{\geq 0}$. We have to show that $\mathscr{P}_{\mathcal{H}_m} \geq \mathscr{P}_{\mathcal{H}_{m+1}}$.

Let $\tau = \varphi_{t_{m+1}}$. Notice that $|\Phi_m(w)| = |\eta_m(w)|$ for any word $w \in \{a, b\}^*$. In particular, $|\Phi_{m+1}(\sigma)| = |\eta_m(\tau(\sigma))|$ for $\sigma \in \{a, b\}$. This implies that $I_{\eta_m \tau} = I_{\Phi_{m+1}} = \eta_{m+1}$, and so $\mathcal{H}_{m+1} = I_{\eta_m \tau}(\mathcal{W}[\alpha_{m+1}])$. Using Proposition 4, we obtain that $\mathscr{P}_{\mathcal{H}_{m+1}} = \mathscr{P}_{I_{\eta_m \tau}(\mathcal{W}[\alpha_{m+1}])} \leq \mathscr{P}_{\eta_m \tau(\mathcal{W}[\alpha_{m+1}])} = \mathscr{P}_{\eta_m(\mathcal{W}[\alpha_m])} = \mathscr{P}_{\mathcal{H}_m}$. □

Observation 2. *If $m > 0$ and $y \in \mathcal{A}_i^m$ or $y \in \mathcal{B}_i^m$, then $\mathcal{W}(y) = \mathcal{W}(i)$.*

Proof. The first part follows directly from the fact that $\mathscr{P}_{\mathcal{H}_m} \leq \mathscr{P}_{\mathcal{H}_0} = \{\mathcal{A}_0^0, \mathcal{B}_0^0\}$ and the fact that $y, i \in \mathcal{A}_i^m$. For the second part, notice that both $W_m^{t_{m+1}} W_{m-1}$, W_{m-1} are prefixes of $\mathcal{W}$. Therefore, $\mathcal{W}(i) = \mathcal{W}(i + t_{m+1} p_m)$ and the claim follows since $i + t_{m+1} p_m \in \mathcal{B}_i^m$. □

6 Shifts in $\mathcal{W}$

As we saw in Section 4.2, we have to examine the set $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$. We start with a simpler task: examining the set $\mathcal{A}_0^0 \,\Delta\, (\mathcal{A}_0^0 \dot{-} x)$, where Δ denotes the symmetric difference. This is the set of y's for which $\mathcal{W}(y) \neq \mathcal{W}(y + x)$.

Notice that $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \,\Delta\, (\mathcal{A}_0^0 \dot{-} x))$.

Our goal is to represent $\mathcal{A}_0^0 \,\Delta\, (\mathcal{A}_0^0 \dot{-} x)$ using the basic sets $\mathcal{A}_i^m$ (for these sets we already have an explicit formula – Proposition 5).

We start with $x = p_k$ for $k \in \mathbb{Z}_{\geq 0}$ and then we generalize to an arbitrary $x \in \mathbb{Z}_{\geq 1}$.

6.1 Shifts by p_k, $k \in \mathbb{Z}_{\geq 0}$

Lemma 3. *Let $k \in \mathbb{Z}_{\geq 0}$. If $0 \leq i < p_{k+1} - 2$, then $\mathcal{W}(i) = \mathcal{W}(i + p_k)$. On the other hand, if $p_{k+1} - 2 \leq i < p_{k+1}$, then $\mathcal{W}(i) \neq \mathcal{W}(i + p_k)$.*

Proof. Notice that $W_{k+1}W_k$ is a prefix of $\mathcal{W}$. By Proposition 2(e), $(W_kW_{k+1})^b$ is also a prefix. This implies the first part. The second part follows from Proposition 2(f). $\square$

The following proposition describes the set $\mathcal{A}_0^0 \,\Delta\, (\mathcal{A}_0^0 \dot{-} p_k)$. It follows from the previous lemma and the fact that $\mathcal{H}_{k+1}$ consists of the blocks $a_0a_1 \cdots a_{p_{k+1}-1}$, $b_0b_1 \cdots b_{p_k-1}$.

Proposition 7. *For* $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{A}_0^0 \,\Delta\, (\mathcal{A}_0^0 \dot{-} p_k) = \mathcal{A}_{p_{k+1}-1}^{k+1} \mathbin{\dot\cup} \mathcal{A}_{p_{k+1}-2}^{k+1}$.

6.2 Arbitrary $x \in \mathbb{Z}_{\geq 1}$

To answer the question for an arbitrary x, we will use the following idea: A generalization of Zeckendorf sums (see [13], [5], [6]) can be used to represent x as a sum of elements from the set $\Pi := \{p_0, p_1, p_2, \ldots\}$. Then, we use Proposition 7 for each of the summands.

Apply the following algorithm on x: While $x \neq 0$, find the largest k such that $p_k \leq x$ and subtract p_k from x. Formally, define two sequences:

$$\begin{aligned} x_0 &:= x, \\ k_i &:= \max\{k \in \mathbb{Z}_{\geq 0} : p_k \leq x_{i-1}\} \quad (i \geq 1), \\ x_i &:= x_{i-1} - p_{k_i} \quad (i \geq 1). \end{aligned}$$

Notice that if $x_i = 0$ for some i, then the two sequences k_j, x_j are not defined for $j > i$. Denote this i by n. Observe that we get a representation of x as a sum of elements from Π: $x = p_{k_1} + p_{k_2} + \cdots + p_{k_n}$.

Example 6. Consider the case $\alpha = [1; 1, 2, 3, \ldots]$, $\Pi = \{1, 2, 5, 17, 73, \ldots\}$, $x = 12 = 5 + 5 + 2$. Here the algorithm yields:

i	0	1	2	3
x_i	12	7	2	0
k_i		2	2	1
p_{k_i}		5	5	2

Let $1 \leq i \leq n$. Denote $\mathcal{X}_i := (\mathcal{A}_0^0 \dot{-} x_{i-1}) \,\Delta\, (\mathcal{A}_0^0 \dot{-} x_i)$ and observe that $\mathcal{A}_0^0 \,\Delta\, (\mathcal{A}_0^0 \dot{-} x) = \mathcal{X}_1 \,\Delta\, \mathcal{X}_2 \,\Delta\, \cdots \,\Delta\, \mathcal{X}_n$. Proposition 7 implies that

$$\mathcal{X}_i = (\mathcal{A}_0^0 \,\Delta\, (\mathcal{A}_0^0 \dot{-} p_{k_i})) \dot{-} x_i = (\mathcal{A}_{p_{k_i+1}-1}^{k_i+1} \mathbin{\dot\cup} \mathcal{A}_{p_{k_i+1}-2}^{k_i+1}) \dot{-} x_i.$$

The fact that $x_i = x_{i-1} - p_{k_i} \le p_{k_i+1} - 1 - p_{k_i} \le p_{k_i+1} - 2$ and Observation 1 imply that $\mathcal{X}_i = \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-1} \cup \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-2}$. Therefore,

$$\mathcal{A}^0_0 \,\Delta\, (\mathcal{A}^0_0 \dot{-} x) = \mathop{\Delta}_{i=1}^{n} (\mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-1} \cup \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-2}).$$

Example 7. For the case in the previous example, we get:

$$\mathcal{A}^0_0 \,\Delta\, (\mathcal{A}^0_0 \dot{-} 12) = (\mathcal{A}^3_9 \cup \mathcal{A}^3_8) \,\Delta\, (\mathcal{A}^3_{14} \cup \mathcal{A}^3_{13}) \,\Delta\, (\mathcal{A}^2_4 \cup \mathcal{A}^2_3).$$

7 The set $\mathcal{B}^0_0 \cap (\mathcal{A}^0_0 \dot{-} x)$

For $x = 1$, since each b of $\mathcal{W}$ is followed by an a, $\mathcal{B}^0_0 \subseteq (\mathcal{A}^0_0 \dot{-} 1)$ and so $\mathcal{B}^0_0 \cap (\mathcal{A}^0_0 \dot{-} 1) = \mathcal{B}^0_0 = \mathcal{A}^1_{t_1}$.

We now assume $x > 1$. Notice that $\mathcal{B}^0_0 \cap (\mathcal{A}^0_0 \dot{-} x) = \mathcal{B}^0_0 \cap \big[\mathcal{A}^0_0 \,\Delta\, (\mathcal{A}^0_0 \dot{-} x)\big]$. Continue with the notation of the previous section. We have:

$$\mathcal{B}^0_0 \cap (\mathcal{A}^0_0 \dot{-} x) = \mathop{\Delta}_{i=1}^{n} \big[(\mathcal{B}^0_0 \cap \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-1}) \cup (\mathcal{B}^0_0 \cap \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-2})\big].$$

Observation 2 implies that $\mathcal{B}^0_0 \cap \mathcal{A}^m_i$ is $\mathcal{A}^m_i$ if $\mathcal{W}(i) = b$ and $\emptyset$ otherwise. We now investigate $\mathcal{W}(p_{k_i+1} - x_i - z)$ for $z \in \{1, 2\}$.

Observation 3. *If $x_i - z' \ge 0$ for $z' \in \{1, 2\}$, then $\mathcal{W}(x_i - z') = \mathcal{W}(x - z')$.*

Proof. By induction on i:

The claim holds trivially for $i = 0$.

For $i > 0$, if $x_i - z' \ge 0$ then also $x_{i-1} - z' \ge 0$. Notice that $x_{i-1} - z' = (x_i - z') + p_{k_i}$ and $x_i - z' \le x_i - 1 \le x_{i-1} - 2 < p_{k_i+1} - 2$. By Lemma 3 and the induction hypothesis, $\mathcal{W}(x_i - z') = \mathcal{W}(x_{i-1} - z') = \mathcal{W}(x - z')$. □

Observation 4. *If $x_i + z \ge 3$ for $z \in \{1, 2\}$, then $\mathcal{W}(p_{k_i+1} - x_i - z) = \mathcal{W}(x + z - 3)$.*

Proof. Proposition 2(g) implies that $\mathcal{W}(p_{k_i+1} - x_i - z) = \mathcal{W}(x_i + z - 3)$ and by the last observation (for $z' = 3 - z$), we get: $\mathcal{W}(p_{k_i+1} - x_i - z) = \mathcal{W}(x + z - 3)$. □

We now consider three cases: (1) $\mathcal{W}(x - 1) = b$, (2) $\mathcal{W}(x - 2) = b$ and (3) $\mathcal{W}(x - 1) = \mathcal{W}(x - 2) = a$.

Consider the first case: For $1 \le i < n$ we have $x_i \ge 1$ and by Observation 4,

$$\mathcal{W}(p_{k_i+1} - x_i - 2) = \mathcal{W}(x-1) = b.$$

Notice that $b = \mathcal{W}(x-1) = \mathcal{W}(x_{n-1}-1) = \mathcal{W}(p_{k_n}-1)$. This means that $2 \nmid k_n$ (see Proposition 2(f)). Therefore, $\mathcal{W}(p_{k_n+1} - x_n - 2) = \mathcal{W}(p_{k_n+1} - 2) = b$.

Hence, for $1 \le i \le n$, $\mathcal{W}(p_{k_i+1} - x_i - 2) = b$. Since $\mathcal{W}$ does not contain bb as a factor, we get that $\mathcal{W}(p_{k_i+1} - x_i - 1) = a$. This implies

$$\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathop{\triangle}_{i=1}^{n} \mathcal{A}_{p_{k_i+1}-x_i-2}^{k_i+1}.$$

The other cases are analyzed similarly. Formulas for the x's of each case can be obtained by considering the blocks of $\mathcal{H}_1$. The following table summarizes the three cases.

Case	$\mathcal{W}(x-2), \mathcal{W}(x-1)$	$x - 2 \in$	$\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$
1	a, b	$\mathcal{A}_{t_1-1}^1$	$\triangle_{i=1}^n \mathcal{A}_{p_{k_i+1}-x_i-2}^{k_i+1}$
2	b, a	$\mathcal{A}_{t_1}^1 = \mathcal{B}_0^0$	$\triangle_{i=1}^n \mathcal{A}_{p_{k_i+1}-x_i-1}^{k_i+1}$
3	a, a	$\mathcal{A}_i^1$ $(i < t_1 - 1)$, $\mathcal{B}_0^1 = \mathcal{A}_{(t_1+1)t_2}^2$	$\mathcal{A}_{t_1}^1 = \mathcal{B}_0^0$

Example 8. For the case described in Example 7, we have $\mathcal{W}(12-1) = b$ and therefore this is Case 1. This implies $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} 12) = \mathcal{A}_8^3 \triangle \mathcal{A}_{13}^3 \triangle \mathcal{A}_3^2$.

8 $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$ as a disjoint union of basic sets

Our goal now is to represent $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$ as a disjoint union of sets of the form $\mathcal{A}_i^m$, instead of taking their symmetric difference as we did in Section 7. Such a representation seems to be much better. However, in order to attain this, we will have to understand better the structure formed by the sets $\mathcal{A}_i^m$, $\mathcal{B}_i^m$.

8.1 The structure of $\mathcal{A}_i^m$, $\mathcal{B}_i^m$

Notice that $\mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m]) = \eta_m\varphi_{t_{m+1}}(\mathcal{W}[\alpha_{m+1}])$, so both $\mathcal{H}_m$, $\mathcal{H}_{m+1}$ consist of blocks of lengths p_{m+1}, p_m in an order determined by $\mathcal{W}[\alpha_{m+1}]$.

By considering these blocks we obtain:

$$\mathcal{A}_i^m = \mathcal{A}_i^{m+1} \mathbin{\dot\cup} \mathcal{A}_{i+p_m}^{m+1} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathcal{A}_{i+(t_{m+1}-1)p_m}^{m+1} \mathbin{\dot\cup} \mathcal{B}_i^{m+1}, \quad \mathcal{B}_i^m = \mathcal{A}_{i+t_{m+1}p_m}^{m+1}.$$

Therefore,

$$\mathcal{A}_i^m = \mathcal{A}_i^{m+1} \mathbin{\dot\cup} \mathcal{A}_{i+p_m}^{m+1} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathcal{A}_{i+(t_{m+1}-1)p_m}^{m+1} \mathbin{\dot\cup} \mathcal{A}_{i+t_{m+2}p_{m+1}}^{m+2}. \tag{5}$$

Definition 7. A *partition tree* of a set $C \neq \emptyset$ is a tree, in which every node is a subset of C, the root is C, and for every node A, which is not a leaf, the set of children of A form a partition of A.

Consider the tree of all the sets $\mathcal{A}_i^m \subseteq \mathcal{B}_0^0$, where there is an edge from $\mathcal{A}_i^m$ to each of the sets in the right-hand side of (5). We denote this tree by $\mathscr{T}_\alpha$. Notice that the root of the tree is $\mathcal{A}_{t_1}^1 = \mathcal{B}_0^0$. Let $\mathbf{pr}\, A$ denote the parent of a set A in the tree. If A is the root, we define $\mathbf{pr}\, A := A$. Notice that $\mathscr{T}_\alpha$ is a partition tree.

Example 9. Figure 3 shows the tree $\mathscr{T}_\alpha$ for $\alpha = [1; 1, 2, 3, \ldots]$. For example, $\mathbf{pr}\, \mathcal{A}_{16}^3 = \mathcal{A}_1^1$ and $\mathbf{pr}\, \mathcal{A}_1^3 = \mathcal{A}_1^2$.

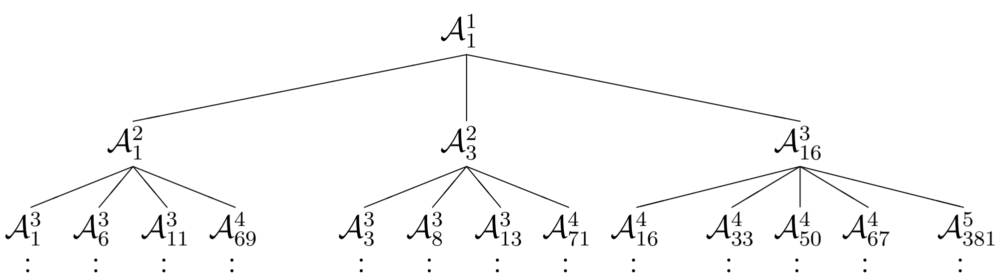


Figure 3: $\mathscr{T}_\alpha$ for $\alpha = [1; 1, 2, 3, \ldots]$

Corollary 1. *Consider the node $\mathcal{A}_i^m$ in $\mathscr{T}_\alpha$, where $\mathcal{A}_i^m$ is not the root. We have*

$$\mathbf{pr}\, \mathcal{A}_i^m = \mathcal{A}_{i \bmod p_{m-1}}^{\overline{m}}, \quad \textit{where} \quad \overline{m} = \begin{cases} m-1, & i < p_{m-1} \cdot t_m \\ m-2, & i \geq p_{m-1} \cdot t_m \end{cases}.$$

Proof. This follows directly from (5). □

8.2 The Chain Proposition

Notice that for Case 3 (see table on page 16) we have $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathcal{A}_{t_1}^1$. So we focus on the first two cases. Let $Z = 2$ for Case 1, and $Z = 1$ for Case 2. Denote $r_i := p_{k_i+1} - x_i - Z$. Then, $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \triangle_{i=1}^n \mathcal{A}_{r_i}^{k_i+1}$.

Proposition 8. *For $1 \leq i < n$, $\mathbf{pr}\,\mathcal{A}_{r_i}^{k_i+1} \subseteq \mathbf{pr}\,\mathcal{A}_{r_{i+1}}^{k_{i+1}+1}$.*

In order to prove Proposition 8 we first prove the following two lemmas:

Lemma 4. *Let $1 \leq k \leq m$, $m \equiv k \pmod 2$, $1 \leq i \leq p_k$. Then, $\mathcal{A}_{p_m-i}^m \subseteq \mathcal{A}_{p_k-i}^k$.*

Proof. By Equation (5), we have that $\mathcal{A}_{p_k-i}^k \supseteq \mathcal{A}_{p_{k+1}\cdot t_{k+2}+(p_k-i)}^{k+2} = \mathcal{A}_{p_{k+2}-i}^{k+2}$. Similarly, $\mathcal{A}_{p_{k+2}-i}^{k+2} \supseteq \mathcal{A}_{p_{k+4}-i}^{k+4}$ and we get the following sequence:

$$\mathcal{A}_{p_k-i}^k \supseteq \mathcal{A}_{p_{k+2}-i}^{k+2} \supseteq \mathcal{A}_{p_{k+4}-i}^{k+4} \supseteq \cdots .$$

Clearly $\mathcal{A}_{p_m-i}^m$ is one of the elements of this sequence and so $\mathcal{A}_{p_m-i}^m \subseteq \mathcal{A}_{p_k-i}^k$. □

Lemma 5. *Let $k \geq 2$, $0 \leq i < p_k - p_{k-1}$. If both $\mathcal{A}_i^k$, $\mathcal{A}_{i+p_{k-1}}^k$ are nodes of $\mathscr{T}_\alpha$, then $\mathbf{pr}\,\mathcal{A}_i^k \subseteq \mathbf{pr}\,\mathcal{A}_{i+p_{k-1}}^k$.*

Proof. Corollary 1 implies that $\mathbf{pr}\,\mathcal{A}_i^k = \mathcal{A}_j^{k_1}$, $\mathbf{pr}\,\mathcal{A}_{i+p_{k-1}}^k = \mathcal{A}_j^{k_2}$ for some j, where $k_1, k_2 \in \{k-1, k-2\}$. Since $i < i + p_{k-1}$, we have $k_2 \leq k_1$.

If $k_1 = k_2$, then the claim holds. Otherwise, $k_1 = k-1$, $k_2 = k-2$. This implies $j < p_{k-2}$, and so $\mathbf{pr}\,\mathcal{A}_i^k = \mathcal{A}_j^{k-1} \subseteq \mathbf{pr}\,\mathcal{A}_j^{k-1} = \mathcal{A}_j^{k-2} = \mathbf{pr}\,\mathcal{A}_{i+p_{k-1}}^k$. □

Proof of Proposition 8. We use the following notation:

$$a := x_i + Z, \quad k := k_i + 1,$$
$$b := x_{i+1} + Z, \quad \ell := k_{i+1} + 1.$$

In this notation, we have to show: $\mathbf{pr}\,\mathcal{A}_{p_k-a}^k \subseteq \mathbf{pr}\,\mathcal{A}_{p_\ell-b}^\ell$.

We have $p_{\ell-1} < a \leq p_\ell + 1$ and $p_\ell - b = p_\ell + p_{\ell-1} - a$. Note that all the sets that are mentioned in the proof are subsets of $\mathcal{B}_0^0$ and therefore they are nodes in $\mathscr{T}_\alpha$.

Consider the following 4 cases: (a) $\ell = 1$, (b) $k \equiv \ell+1 \pmod 2$, (c) $a \leq p_\ell$ and $k \equiv \ell \pmod 2$, (d) $a = p_\ell + 1$ and $k \equiv \ell \pmod 2$.

(a) is trivial. We show here the proof of (c). (b), (d) are proven similarly using applications of Lemma 4, Lemma 5 and Corollary 1.

Suppose that $a \le p_\ell$ and $k \equiv \ell \pmod 2$. Lemma 4 implies that $\mathcal{A}^k_{p_k - a} \subseteq \mathcal{A}^\ell_{p_\ell - a}$. Therefore, $\mathbf{pr}\,\mathcal{A}^k_{p_k-a} \subseteq \mathbf{pr}\,\mathcal{A}^\ell_{p_\ell-a}$. Lemma 5 implies that

$$\mathbf{pr}\,\mathcal{A}^k_{p_k-a} \subseteq \mathbf{pr}\,\mathcal{A}^\ell_{p_\ell-a} \subseteq \mathbf{pr}\,\mathcal{A}^\ell_{p_{\ell-1}+p_\ell-a} = \mathbf{pr}\,\mathcal{A}^\ell_{p_\ell-b}. \qquad \square$$

8.3 A disjoint union

Proposition 8 implies that the sets that participate in the symmetric difference satisfy the following property:

$$\mathbf{pr}\,\mathcal{A}^{k_1+1}_{r_1} \subseteq \mathbf{pr}\,\mathcal{A}^{k_2+1}_{r_2} \subseteq \mathbf{pr}\,\mathcal{A}^{k_3+1}_{r_3} \subseteq \cdots \subseteq \mathbf{pr}\,\mathcal{A}^{k_n+1}_{r_n}. \tag{6}$$

Theorem 2. *The set $\mathcal{B}^0_0 \cap (\mathcal{A}^0_0 \dot{-} x)$ can be written as a disjoint union of $O(\Sigma_{i=1}^{k_1+1} t_i)$ sets of the form $\mathcal{A}^m_i$.*

Notice that if $t_i < T$ for all $i \in \mathbb{Z}_{\ge 1}$, then the number of sets is $O(T \log x)$.

Proof. Define a partition subtree to be a subtree which is also a partition tree. In other words, every node of the subtree which is not a leaf, should have the same set of children as the same node in the original partition tree.

Consider the minimal partition subtree of $\mathscr{T}_\alpha$ that contains the node $\mathcal{A}^{k_1+1}_{r_1}$. Denote it by T_x. This tree consists of the nodes $\mathbf{pr}^i\,\mathcal{A}^{k_1+1}_{r_1}$ $(i \in \mathbb{Z}_{\ge 1})$ and their children. Notice that (6) guarantees that all the sets $\mathcal{A}^{k_i+1}_{r_i}$ are nodes in the tree. The tree has at most k_1+1 layers, so the number of nodes is at most $\sum_{i=1}^{k_1+1}(t_i+1)$. It is easy to see that in every finite partition tree, each element of the algebra (of sets) generated by the nodes, is a disjoint union of leaves. $\square$

Notice that Theorem 2 can be used to write an algorithm that gets x and outputs a list of sets $\mathcal{A}^m_i$ whose disjoint union is $\mathcal{B}^0_0 \cap (\mathcal{A}^0_0 \dot{-} x)$: Compute the tree T_x and mark the sets $\mathcal{A}^{k_i+1}_{r_i}$ in it. Visit the nodes of the tree, starting from the root, and if an internal node is marked, replace its mark with its children. Then, output the marked leaves.

Example 10. Consider the sets that appear in Example 8. The minimal partition subtree that contains $\mathcal{A}^3_8$ is shown in Figure 4. We have $\mathcal{B}^0_0 \cap (\mathcal{A}^0_0 \dot{-} 12) = \mathcal{A}^3_8 \,\Delta\, \mathcal{A}^3_{13} \,\Delta\, \mathcal{A}^2_3 = \mathcal{A}^3_3 \cup \mathcal{A}^4_{71}$.

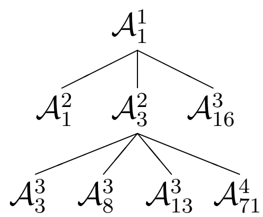


Figure 4: T_{12}

9 E_x as a union of basic sets

We saw that $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \dot{\bigcup}_{j=1}^{n'} \mathcal{A}_{i_j}^{m_j}$ for some $n', i_1, m_1, \ldots, i_{n'}, m_{n'}$. Proposition 3 implies that $E_x = \bigcup_{j=1}^{n'} F[\mathcal{A}_{i_j}^{m_j}]$ where $F(s) = hg^{-1}(s+x+1) - gh^{-1}(s+1)$. In this section we give a somewhat better representation of E_x.

9.1 The general case

We start by computing $h^{-1}(s+1)$ for $s \in \mathcal{A}_i^m \subseteq \mathcal{B}_0^0$. Suppose that s is the n-th element of $\mathcal{A}_i^m$. It is generated (when applying Φ_m) by the n-th a of $\mathcal{W}[\alpha_m]$. Let $j = h^{-1}(i+1)$ be the number of b's in the first $i+1$ letters of $\Phi_m(a)$. Since the n-th a of $\mathcal{W}[\alpha_m]$ is in position $\lfloor \alpha_m n \rfloor - 1$, there are $(n-1)$ a's and $(\lfloor \alpha_m n \rfloor - n)$ b's before this a. Each a contributes (when applying Φ_m) $(p_m - q_m)$ b's and each b contributes $(p_{m-1} - q_{m-1})$ b's. This implies:

$$h^{-1}(s+1) = (p_m - q_m) \cdot (n-1) + (p_{m-1} - q_{m-1}) \cdot (\lfloor \alpha_m n \rfloor - n) + j.$$

In other words, there are constants $A, B, C \in \mathbb{Z}$ such that $h^{-1}(s+1) = A\lfloor \alpha_m n \rfloor + Bn + C$.

In order to compute $g^{-1}(s+x+1)$ we will need the following generalization of a proposition that appears in [3] (it is proven there for the case $\alpha = [1; 1, 1, \ldots]$):

Proposition 9. *If bua is a factor of $\mathcal{W}$ where $n = |u|$ then $|u|_a = |w|_a$ and $|u|_b = |w|_b$ where w is the prefix of $\mathcal{W}$ of length n.*

Proof. It suffices to prove that $|u|_b = |w|_b$ as $|u| = |w|$. Denote by j the index of the first b of the bua factor.

Let $X = \{i\beta : i \in \mathbb{Z}\}$. Notice that $(z+1, z+2) \cap X \neq \emptyset$ if and only if $\mathcal{W}(z) = b$. Let $f : \mathbb{R} \to \mathbb{Z}$, $f(x) = |(x, x+n) \cap X|$. In other words, $f(x)$ is the number of points from X in the interval $(x, x+n)$. It is easy to see that f is periodic with period β and that f is increasing on the interval $[0, \beta)$.

Notice that $|u|_b = f(j+2)$ and $|w|_b = f(1)$. Since we have an a after the u it implies that $f(j+3) \leq f(j+2)$. We also know that there is a b before the u and therefore there is $r \in \mathbb{Z}$ such that $j+1 < \beta r < j+2$. Hence

$$\beta r < j+2 < \beta r + 1 < j+3 < \beta(r+1).$$

But f is increasing in the interval $[\beta r, \beta(r+1))$ and so

$$f(j+2) \leq f(\beta r + 1) \leq f(j+3) \leq f(j+2).$$

We conclude that $|w|_b = f(1) = f(\beta r + 1) = f(j+2) = |u|_b$. □

Notice that $\mathcal{W}(s-1) = a$. We can give a formula for $g^{-1}(s)$ in a similar way to what we did for $h^{-1}(s+1)$. Let w be the prefix of length $x-1$. By the last proposition, we have $g^{-1}(s+x+1) = g^{-1}(s) + |w|_a + 1$ and so we get a formula for $g^{-1}(s+x+1)$ that has the form $A' \lfloor \alpha_m n \rfloor + B'n + C'$.

We conclude that the set E_x can be written as a union of sets of the form

$$\{h\left(A' \lfloor \alpha_m n \rfloor + B'n + C'\right) - g\left(A \lfloor \alpha_m n \rfloor + Bn + C\right) : n \in \mathbb{Z}_{\geq 1}\},$$

where $A, B, C, A', B', C' \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$.

Example 11. For $\alpha = [1; 1, 2, 3, \ldots]$ we have $E_{12} = F[\mathcal{A}_3^3] \cup F[\mathcal{A}_{71}^4]$ and

$$F[\mathcal{A}_3^3] = \{h(3\lfloor \alpha_3 n \rfloor + 7n) - g(2\lfloor \alpha_3 n \rfloor + 5n - 5) : n \in \mathbb{Z}_{\geq 1}\},$$
$$F[\mathcal{A}_{71}^4] = \{h(10\lfloor \alpha_4 n \rfloor + 33n + 7) - g(7\lfloor \alpha_4 n \rfloor + 23n) : n \in \mathbb{Z}_{\geq 1}\},$$
$$\alpha_3 = [1; 4, 5, 6, \ldots] \approx 1.23845, \quad \alpha_4 = [1; 5, 6, 7, \ldots] \approx 1.19369.$$

9.2 The case $\alpha = [1; t, t, t, \ldots]$

In turns out that in the case $\alpha = [1; t, t, t, \ldots]$ there is a simpler relation between E_x and $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$:

Proposition 10. *Let $x \in \mathbb{Z}_{\geq 1}$. There exists $C \in \mathbb{Z}$ such that $F(s) = ts + C$ for any $s \in \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$.*

Proof. Let $s \in \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$. Notice that $\beta = \alpha + t$ and so $h(y) = g(y) + yt$. Therefore, $h(g^{-1}(s+x+1)) = g^{-1}(s+x+1)t + s + x + 1$ and $g(h^{-1}(s+1)) = s + 1 - h^{-1}(s+1)t$. We also have $h^{-1}(s+1) + g^{-1}(s) = s+1$. This implies, $F(s) = hg^{-1}(s+x+1) - gh^{-1}(s+1) = x + \left[g^{-1}(s+x+1) - g^{-1}(s) + s + 1\right] t$. Proposition 9 implies that $g^{-1}(s+x+1) - g^{-1}(s)$ does not depend on s and this completes the proof. □

10 Conclusion

We saw that the maximal set of moves that defines a game with P-positions $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$ is $\mathbb{V} \setminus (\mathscr{M}_1 \cup \mathscr{M}_2)$. We represented this set by a matrix (a_{xy}) where a_{xy} indicates whether $(x, y) \in \mathscr{M}_1$ and whether $(x, y) \in \mathscr{M}_2$.

We examined the structure of any fixed row, x, of this matrix. The set $\mathscr{M}_1$ may contribute at most 4 elements for each row. We gave a description of $\mathscr{M}_1$ that facilitates computing these elements. For the set $\mathscr{M}_2$, we defined $E_x = \{y \geq x : (x, y) \in \mathscr{M}_2\}$. We saw that E_x is related to the α-word in the following manner: $E_x = F[\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)]$ where $F(s) = hg^{-1}(s + x + 1) - gh^{-1}(s + 1)$.

The next step was to investigate the set $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$. In order to do it, we wrote x as a sum of p_i's. In the process, we obtained two sequences: $x = x_0 > x_1 > \ldots > x_n = 0$ and $k_1 \geq k_2 \geq \ldots \geq k_n$, such that $\sum_{j=i+1}^{n} p_{k_j} = x_i$. It turned out that there are 3 cases:

1. When $\mathcal{W}(x - 1) = b$, we have $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \triangle_{i=1}^{n} \mathcal{A}_{p_{k_i+1} - x_i - 2}^{k_i+1}$.
2. When $\mathcal{W}(x - 2) = b$, we have $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \triangle_{i=1}^{n} \mathcal{A}_{p_{k_i+1} - x_i - 1}^{k_i+1}$.
3. When $\mathcal{W}(x - 1) = \mathcal{W}(x - 2) = a$, we have $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathcal{B}_0^0 = \mathcal{A}_{t_1}^1$.

For the first two cases, we provided an algorithm that converts the symmetric difference to a disjoint union of sets of the form $\mathcal{A}_i^m$.

Then we showed a way to simplify $F[\mathcal{A}_i^m]$, and we concluded that E_x is the union of sets of the form

$$\{h\left(A' \lfloor \alpha_m n \rfloor + B'n + C'\right) - g\left(A \lfloor \alpha_m n \rfloor + Bn + C\right) : n \in \mathbb{Z}_{\geq 1}\}.$$

Examples 6, 7, 8, 10, 11 show the process for the case $\alpha = [1; 1, 2, 3, \ldots]$ and $x = 12$.

10.1 Further directions of research

10.1.1 Zeckendorf sums

Let $x \in \mathbb{Z}_{\geq 0}$. It is well known (see, for example, [6] and [5]) that x can be written as $x = \sum_{i=0}^{\infty} \tilde{x}_i p_i$ where $0 \leq \tilde{x}_i \leq t_{i+1}$ such that if $\tilde{x}_i = t_{i+1}$ for some $i > 0$ then $\tilde{x}_{i-1} = 0$. Moreover, this representation is unique.

Definition 8. For $x \in \mathbb{Z}_{\geq 0}$, define $R_m(x) = \sum_{i=0}^{m-1} \tilde{x}_i p_i$.

The following proposition, which we do not prove here, gives another definition for the sets $\mathcal{A}_i^m$, $\mathcal{B}_i^m$:

Proposition 11. *$\mathcal{A}_i^m = \{x \in \mathbb{Z}_{\geq 0} : R_m(x) = i$ and $\tilde{x}_m < t_{m+1}\}$ and $\mathcal{B}_i^m = \{x \in \mathbb{Z}_{\geq 0} : R_m(x) = i$ and $\tilde{x}_m = t_{m+1}\}$.*

This definition gives us another way to look at these sets. It is possible that one can rewrite the claims we proved here using the α-word, and use the definition in Proposition 11 instead.

10.1.2 Finding a "nice" set of moves

For generalized Wythoff, we have a "nice" set of moves that defines the game: $\{(0,k) : k \in \mathbb{Z}_{\geq 1}\} \cup \{(k,\ell) : k,\ell \in \mathbb{Z}_{\geq 1}, 0 \leq \ell - k < t\}$. For $\alpha = [1; 1, t, 1, t, \ldots]$ there is also a "nice" set of moves (see [4]). However, for an arbitrary irrational $1 < \alpha < 2$, this is not the case. [8] shows the construction of such a set and here we described the maximal set, but neither can be considered "nice". The question is whether such a "nice" set of moves exists for the case of an arbitrary α or for some subset of the possible α's.

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