CHARACTERIZING THE NUMBER OF m–ARY PARTITIONS WITH NO GAPS MODULO m

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Abstract. In a recent work, the authors provided the first-ever characterization of the values $b_m(n)$ modulo m where $b_m(n)$ is the number of (unrestricted) m-ary partitions of the integer n and $m \geq 2$ is a fixed integer. That characterization proved to be quite elegant and relied only on the base m representation of n. Since then, the authors have been motivated to consider a specific restricted m-ary partition function, namely $c_m(n)$, the number of m-ary partitions of n where there are no "gaps" in the parts. (That is to say, if m^i is a part in a partition counted by $c_m(n)$, and i is a positive integer, then m^{i-1} must also be a part in the partition.) Using tools similar to those utilized in the aforementioned work on $b_m(n)$, we prove the first-ever characterization of $c_m(n)$ modulo m. As with the work related to $b_m(n)$ modulo m, this characterization of $c_m(n)$ modulo m is also based solely on the base m representation of n.

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1. Introduction

In this note, we will focus our attention on congruence properties for the partition functions which enumerate restricted integer partitions known as m-ary partitions. These are partitions of an integer n wherein each part is a power of a fixed integer $m \geq 2$. Throughout this note, we will let $b_m(n)$ denote the number of m-ary partitions of n.

As an example, note that there are five 3-ary partitions of $n = 9$:

$$9, \quad 3+3+3, \quad 3+3+1+1+1,$$

$$3+1+1+1+1+1+1, \quad 1+1+1+1+1+1+1+1+1$$

Thus, $b_3(9) = 5$.

In the late 1960s, Churchhouse [5, 6] initiated the study of congruence properties of binary partitions (m-ary partitions with $m = 2$). Within months, other mathematicians proved Churchhouse's conjectures and proved natural extensions of his results. These included Rødseth [9] who extended Churchhouse's results to include the functions $b_p(n)$ where p is any prime as well as Andrews [2] and Gupta [7, 8] who proved that corresponding results also held for $b_m(n)$ where m could be any integer greater than 1. As part of an infinite family of results, these authors proved that, for any $m \geq 2$ and any nonnegative integer n, $b_m(m(mn-1)) \equiv 0 \pmod{m}$.

Quite recently, the authors [3] provided the following mod m characterization of $b_m(mn)$ relying solely on the base m representation of n:

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Theorem 1.1. *If $m \geq 2$ is a fixed integer and*

$$n = \alpha_0 + \alpha_1 m + \cdots + \alpha_j m^j$$

is the base m representation of n (so that $0 \leq \alpha_i \leq m-1$ for each i), then

$$b_m(mn) \equiv \prod_{i=0}^{j} (\alpha_i + 1) \pmod{m}.$$

In this note, we provide a similar mod m result for the values $c_m(mn)$, where $c_m(n)$ is the number of m–ary partitions of n with "no gaps" in the parts. More specifically, $c_m(n)$ counts the number of partitions of n into powers of m such that, if m^i is a part in a partition counted by $c_m(n)$, and i is a positive integer, then m^{i-1} must also be a part in the partition. For example, there are six such partitions counted by $c_3(15)$:

$$9+3+1+1+1, \quad 3+3+3+3+1+1+1, \quad 3+3+3+1+1+1+1+1+1,$$

$$3+3+1+1+1+1+1+1+1+1+1, \quad 3+1+1+1+1+1+1+1+1+1+1+1+1,$$

$$1+1+1+1+1+1+1+1+1+1+1+1+1+1+1$$

Note, in particular, that $9+1+1+1+1+1+1$ does not appear in the above list because it does not contain the part 3, and $3+3+3+3+3$ is missing from the list because it does not contain the part 1.

This family of functions $c_m(n)$ is motivated by a recent work of Bessenrodt, Olsson, and Sellers [4] in which the function $c_2(n)$ plays a critical role.

2. The Main Result

The following theorem provides a complete characterization of $c_m(mn)$ modulo m:

Theorem 2.1. *Let $m \geq 2$ be a fixed integer and let*

$$n = \sum_{i=j}^{\infty} \alpha_i m^i$$

be the base m representation of n where $1 \leq \alpha_j < m$ and $0 \leq \alpha_i < m$ for $i > j$.

(1) *If j is even, then*

$$c_m(mn) \equiv \alpha_j + (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \dots \alpha_i \pmod{m}.$$

(2) *If j is odd, then*

$$c_m(mn) \equiv 1 - \alpha_j - (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \dots \alpha_i \pmod{m}.$$

Remark 2.2. *Note that Lemma 2.7 (which appears below) implies that Theorem 2.1 tells us the congruence class of $c_m(n)$ modulo m for all n, not just those values of n which are divisible by m.*

In order to prove Theorem 2.1, we need a few elementary tools. We describe these tools here.

First, it is important to note the generating function for $c_m(n)$.

Lemma 2.3.

$$C_m(q) := 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+m^2+\cdots+m^n}}{(1-q)(1-q^m)\dots(1-q^{m^n})}.$$

Proof. The proof follows from a standard argument from [1, Chapter 1]. ∎

Next, we wish to find the generating function for $c_m(mn)$.

Lemma 2.4.

$$\sum_{n=0}^{\infty} c_m(mn)q^n = 1 + \frac{q}{1-q}C_m(q) \tag{1}$$

Proof. Note that $C_m(q)$ can be rewritten as

$$\begin{aligned} C_m(q) &= 1 + \sum_{n=0}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\dots(1-q^{m^n})}\frac{q}{1-q} \\ &= 1 + \frac{q}{1-q} + \sum_{n=1}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\dots(1-q^{m^n})} \cdot \sum_{j=0}^{\infty} q^j. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} c_m(mn)q^{mn} &= \frac{1}{1-q^m} + \sum_{n=1}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\dots(1-q^{m^n})} \cdot \sum_{j=0}^{\infty} q^{jm} \\ &= \frac{1}{1-q^m} + \frac{q^m}{1-q^m} \cdot \sum_{n=1}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\dots(1-q^{m^n})} \\ &= \frac{1}{1-q^m} + \frac{q^m}{1-q^m}(C_m(q^m) - 1) \\ &= 1 + \frac{q^m}{1-q^m} + \frac{q^m}{1-q^m}C_m(q^m). \end{aligned}$$

The proof follows by replacing q^m by q. ∎

From Lemma 2.4, we have the following recurrence satisfied by $c_m(mn)$.

Lemma 2.5. *For $n \geq 1$,*

$$c_m(mn) = c_m(0) + c_m(1) + \cdots + c_m(n-1).$$

Proof. Compare coefficients of q^n on both sides of the identity in Lemma 2.4. ∎

Lemma 2.6.

$$C_m(q) = -q^{-1} - q^{-2} - \cdots - q^{-(m-1)} + (1 + q^{-1} + \cdots + q^{-(m-1)}) \sum_{n=0}^{\infty} c_m(mn)q^{mn}$$

Proof. By Lemma 2.4,

$$\sum_{n=0}^{\infty} c_m(mn)q^{mn} = 1 + \frac{q^m}{1-q^m}C_m(q^m).$$

On the other hand,

$$\begin{aligned} C_m(q) &= 1 + \frac{q}{1-q} + \sum_{n=1}^{\infty} \frac{q^{m+\ldots m^n}}{(1-q^m)\ldots(1-q^{mn})} \cdot \frac{q}{1-q} \\ &= \frac{1}{1-q} + \frac{q}{1-q} \sum_{n=0}^{\infty} \frac{q^{m(1+m+\ldots m^n)}}{(1-q^m)\ldots(1-q^{m\cdot m^n})}) \\ &= \frac{1}{1-q} + \frac{q}{1-q} C_m(q^m). \end{aligned}$$

Therefore,

$$C_m(q^m) = q^{-1}(C_m(q)(1-q) - 1)$$

and so

$$\sum_{n=0}^{\infty} c_m(mn) q^{mn} = 1 + \frac{q^{m-1}}{1-q^m}(C_m(q)(1-q) - 1).$$

Solving for $C_m(q)$ gives the desired result. ∎

Lemma 2.6 can now be used to prove that the values of the function $c_m(n)$ come in m–tuples as described in the next lemma.

Lemma 2.7. *For all* $n \geq 1$,

$$c_m(mn) = c_m(mn-1) = c_m(mn-2) = \cdots = c_m(mn-(m-1)).$$

Proof. Compare coefficients of q^n on both sides of the identity in Lemma 2.6. ∎

We now begin the consideration of $c_m(mn)$ modulo m by proving the following lemma:

Lemma 2.8. *If* $n \equiv k \pmod{m}$ *where* $1 \leq k \leq m$, *then*

$$c_m(mn) \equiv 1 + (k-1)c_m(n) \pmod{m}.$$

Proof. By Lemma 2.5,

$$c_m(mn) = c_m(0) + c_m(1) \cdots + c_m(n-1).$$

Next, we write $n = jm + k$ for some integer j. Then

$$\begin{aligned} c_m(mn) &= c_m(0) + c_m(1) + \cdots + c_m(m) \\ &\quad + c_m(m+1) + \cdots + c_m(2m) \\ &\quad \vdots \\ &\quad + c_m((j-1)m+1) + \cdots + c_m((j-1)m+m) \\ &\quad + c_m(jm+1) + \cdots + c_m(jm+k-1) \\ &\equiv 1 + c_m(jm+1) + \cdots + c_m(jm+k-1) \pmod{m} \\ &\qquad \text{by Lemma 2.7} \\ &\equiv 1 + (k-1)c_m(jm+k) \pmod{m} \\ &\qquad \text{by Lemma 2.7} \\ &= 1 + (k-1)c_m(n). \end{aligned}$$

∎

Next, we prove an additional lemma involving an "internal" congruence satisfied by c_m modulo m. It is interesting to note that a similar result holds for $b_m(n)$, the unrestricted m-ary partition function studied in [3, 5, 6].

Lemma 2.9. *For all* $n \geq 0$,

$$c_m(m^3n) \equiv c_m(mn) \pmod{m}.$$

Proof. By Lemma 2.8, we know

$$\begin{aligned} c_m(m^3n) &= c_m(m(m^2n)) \\ &\equiv 1 + (m-1)c_m(m^2n) \pmod{m} \\ &= 1 + (m-1)c_m(m(mn)) \\ &\equiv 1 + (m-1)(1 + (m-1)c_m(mn)) \pmod{m} \\ &\equiv c_m(mn) \pmod{m}. \end{aligned}$$

∎

Lemma 2.9 enables a significant reduction in the number of cases which will need to be checked when we prove Theorem 2.1. This is because of the following. Given n written in m-ary notation as

$$n = \alpha m^j + \beta m^k + \cdots + \gamma m^r,$$

we see immediately that

$$mn = \alpha m^{j+1} + \beta m^{k+1} + \cdots + \gamma m^{r+1},$$

where $\alpha, \beta, \dots, \gamma \in \{1, 2, \dots, m-1\}$ and $j < k < \cdots < r$. Thus, we can divide by m^2 for as many times as we wish if $j \geq 2$ (because $j + 1 \geq 3$). Therefore, we only need to consider the cases $j = 0$ and $j = 1$ in what follows.

We are now in a position to prove Theorem 2.1 which provides a characterization of $c_m(mn)$ modulo m simply based on the m–ary representation of n.

Proof. By Lemma 2.9, we see that if $j \geq 2$, then $m^3 \mid mn$. This means $c_m(mn) = c_m\left(\frac{n}{m}\right)$. Thus, we may assume $j = 0$ or $j = 1$ without loss of generality.

Now we consider two cases (based on the parity of j).

- Case 1: j is even, so we can assume $j = 0$. Hence,

$$\begin{aligned} c_m(mn) &\equiv 1 + (\alpha_0 - 1)c_m(n) \pmod{m} \\ &= 1 + (\alpha_0 - 1)c_m(\alpha_0 + \alpha_1 m + \alpha_2 m^2 + \dots). \end{aligned}$$

  Now since $m > \alpha_0 \geq 1$, we may replace α_0 by m (thanks to Lemma 2.7). Then the above becomes

$$\begin{aligned} c_m(mn) &\equiv 1 + (\alpha_0 - 1)c_m((\alpha_1 + 1)m + \alpha_2 m^2 + \dots) \pmod{m} \\ &= 1 + (\alpha_0 - 1)c_m(m((\alpha_1 + 1) + \alpha_2 m + \alpha_3 m^2 + \dots)) \\ &\equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m((\alpha_1 + 1) + \alpha_2 m + \alpha_3 m^2 + \dots)) \pmod{m}. \end{aligned}$$

  Now $1 \leq \alpha_1 + 1 \leq m$, so by Lemma 2.7 we may replace $\alpha_1 + 1$ by m in the above to obtain

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m(m(\alpha_2 + 1) + \alpha_3 m + \dots)) \pmod{m}.$$

Now $1 \leq \alpha_2 + 1 \leq m$, so we may apply Lemma 2.7 again, and the process continues until we hit some $\alpha_i = 0$ at which time the process terminates. The result is

$$\begin{aligned} c_m(mn) &\equiv 1 + (\alpha_0 - 1)(1 + \alpha_1(1 + \alpha_2(1 + \alpha_3 + \dots))) \pmod m \\ &= \alpha_0 + (\alpha_0 - 1)\sum_{i=1}^{\infty} \alpha_1\alpha_2\dots\alpha_i \end{aligned}$$

which is equivalent to the first case of Theorem 2.1.

- Case 2: j is odd, so we can assume $j = 1$. Hence, $n \equiv m \pmod m$, and by Lemma 2.8,

$$\begin{aligned} c_m(mn) &\equiv 1 - c_m(n) \pmod m \\ &= 1 - c_m\left(m\sum_{j=0}^{\infty} \alpha_{j+1}m^j\right). \end{aligned}$$

Now Case 1 above is applicable to $n' = \sum_{j=0}^{\infty} \alpha_{j+1}m^j$ because $1 \leq \alpha_1 < m$. Hence, the desired result follows.

∎

With the goal of demonstrating the applicability of Theorem 2.1, we compute a few examples.

- Let $m = 4$, $n = 123 = 3 + 2 \cdot 4 + 3 \cdot 4^2 + 1 \cdot 4^3$. Then

$$c_4(4 \cdot 123) = c_4(492) = 5843 \equiv 3 \pmod 4.$$

This is an example of the case $j = 0$. Theorem 2.1 asserts that

$$\begin{aligned} c_4(4 \cdot 123) &\equiv 3 + (3 - 1)(2 + 2 \cdot 3 + 2 \cdot 3 \cdot 1) \pmod 4 \\ &= 3 + 2 \cdot 14 \\ &\equiv 3 \pmod 4 \end{aligned}$$

as computed above.

- Let $m = 5$, $n = 485 = 2 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3$. Then

$$c_5(5 \cdot 485) = c_5(2425) = 230358 \equiv 3 \pmod 5.$$

This is an example of the case $j = 1$. Theorem 2.1 asserts that

$$\begin{aligned} c_5(5 \cdot 485) &\equiv 1 - 2 - (2 - 1)(4 + 4 \cdot 3) \pmod 5 \\ &= 1 - 2 - 16 \\ &= -17 \\ &\equiv 3 \pmod 5 \end{aligned}$$

as computed above.

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