CHARACTERIZING THE NUMBER OF $m$–ARY PARTITIONS WITH NO GAPS MODULO $m$

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Abstract. In a recent work, the authors provided the first-ever characterization of the values $b_m(n)$ modulo $m$ where $b_m(n)$ is the number of (unrestricted) $m$-ary partitions of the integer $n$ and $m \geq 2$ is a fixed integer. That characterization proved to be quite elegant and relied only on the base $m$ representation of $n$. Since then, the authors have been motivated to consider a specific restricted $m$-ary partition function, namely $c_m(n)$, the number of $m$-ary partitions of $n$ where there are no “gaps” in the parts. (That is to say, if $m^i$ is a part in a partition counted by $c_m(n)$, and $i$ is a positive integer, then $m^{i-1}$ must also be a part in the partition.) Using tools similar to those utilized in the aforementioned work on $b_m(n)$, we prove the first-ever characterization of $c_m(n)$ modulo $m$. As with the work related to $b_m(n)$ modulo $m$, this characterization of $c_m(n)$ modulo $m$ is also based solely on the base $m$ representation of $n$. 

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1. Introduction

In this note, we will focus our attention on congruence properties for the partition functions which enumerate restricted integer partitions known as $m$-ary partitions. These are partitions of an integer $n$ wherein each part is a power of a fixed integer $m \geq 2$. Throughout this note, we will let $b_m(n)$ denote the number of $m$-ary partitions of $n$.

As an example, note that there are five 3-ary partitions of $n = 9$:

$9$, $3 + 3 + 3$, $3 + 3 + 1 + 1$, $3 + 1 + 1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

Thus, $b_3(9) = 5$.

In the late 1960s, Churchhouse [5, 6] initiated the study of congruence properties of binary partitions ($m$-ary partitions with $m = 2$). Within months, other mathematicians proved Churchhouse’s conjectures and proved natural extensions of his results. These included Rodseth [9] who extended Churchhouse’s results to include the functions $b_p(n)$ where $p$ is any prime as well as Andrews [2] and Gupta [7, 8] who proved that corresponding results also held for $b_m(n)$ where $m$ could be any integer greater than 1. As part of an infinite family of results, these authors proved that, for any $m \geq 2$ and any nonnegative integer $n$, $b_m(mn - 1) \equiv 0 \pmod{m}$.

Quite recently, the authors [3] provided the following mod $m$ characterization of $b_m(mn)$ relying solely on the base $m$ representation of $n$:

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Theorem 1.1. If $m \geq 2$ is a fixed integer and

$$n = \alpha_0 + \alpha_1 m + \cdots + \alpha_j m^j$$

is the base $m$ representation of $n$ (so that $0 \leq \alpha_i \leq m - 1$ for each $i$), then

$$b_m(mn) \equiv \prod_{i=0}^{j} (\alpha_i + 1) \pmod{m}.$$

In this note, we provide a similar mod $m$ result for the values $c_m(mn)$, where $c_m(n)$ is the number of $m$–ary partitions of $n$ with “no gaps” in the parts. More specifically, $c_m(n)$ counts the number of partitions of $n$ into powers of $m$ such that, if $m^i$ is a part in a partition counted by $c_m(n)$, and $i$ is a positive integer, then $m^{i-1}$ must also be a part in the partition. For example, there are six such partitions counted by $c_3(15)$:

- $9 + 3 + 1 + 1 + 1 + 1 + 1,$
- $3 + 3 + 3 + 3 + 1 + 1 + 1,$
- $3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1,$
- $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.$

Note, in particular, that $9 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$ does not appear in the above list because it does not contain the part 3, and $3 + 3 + 3 + 3 + 3$ is missing from the list because it does not contain the part 1.

This family of functions $c_m(n)$ is motivated by a recent work of Bessenrodt, Olsson, and Sellers [4] in which the function $c_2(n)$ plays a critical role.

2. The Main Result

The following theorem provides a complete characterization of $c_m(mn)$ modulo $m$:

Theorem 2.1. Let $m \geq 2$ be a fixed integer and let

$$n = \sum_{i=j}^{\infty} \alpha_i m^i$$

be the base $m$ representation of $n$ where $1 \leq \alpha_j < m$ and $0 \leq \alpha_i < m$ for $i > j$.

1. If $j$ is even, then

$$c_m(mn) \equiv \alpha_j + (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \cdots \alpha_i \pmod{m}.$$

2. If $j$ is odd, then

$$c_m(mn) \equiv 1 - \alpha_j - (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \cdots \alpha_i \pmod{m}.$$

Remark 2.2. Note that Lemma 2.7 (which appears below) implies that Theorem 2.1 tells us the congruence class of $c_m(n)$ modulo $m$ for all $n$, not just those values of $n$ which are divisible by $m$.

In order to prove Theorem 2.1, we need a few elementary tools. We describe these tools here.

First, it is important to note the generating function for $c_m(n)$.
Lemma 2.3.

\[ C_m(q) := 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+m^2+\cdots+m^n}}{(1-q)(1-q^m)\cdots(1-q^{mn})}. \]

Proof. The proof follows from a standard argument from [1, Chapter 1]. □

Next, we wish to find the generating function for \( c_m(mn) \).

Lemma 2.4.

(1) \[ \sum_{n=0}^{\infty} c_m(mn)q^n = 1 + \frac{q}{1-q} C_m(q) \]

Proof. Note that \( C_m(q) \) can be rewritten as

\[ C_m(q) = 1 + \sum_{n=0}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\cdots(1-q^{mn})} \frac{q}{1-q} \]

\[ = 1 + \frac{q}{1-q} + \sum_{n=1}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\cdots(1-q^{mn})} \cdot \sum_{j=0}^{\infty} q^j. \]

Hence,

\[ \sum_{n=0}^{\infty} c_m(mn)q^{mn} = \frac{1}{1-q^m} + \sum_{n=1}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\cdots(1-q^{mn})} \cdot \sum_{j=0}^{\infty} q^{jm} \]

\[ = \frac{1}{1-q^m} + \frac{q^m}{1-q^m} \cdot \sum_{n=1}^{\infty} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\cdots(1-q^{mn})} \]

\[ = \frac{1}{1-q^m} + \frac{q^m}{1-q^m} (C_m(q^m) - 1) \]

\[ = 1 + \frac{q^m}{1-q^m} + \frac{q^m}{1-q^m} C_m(q^m). \]

The proof follows by replacing \( q^m \) by \( q \). □

From Lemma 2.4, we have the following recurrence satisfied by \( c_m(mn) \).

Lemma 2.5. For \( n \geq 1 \),

\[ c_m(mn) = c_m(0) + c_m(1) + \cdots + c_m(n-1). \]

Proof. Compare coefficients of \( q^n \) on both sides of the identity in Lemma 2.4. □

Lemma 2.6.

\[ C_m(q) = -q^{-1} - q^{-2} - \cdots - q^{-(m-1)} + (1 + q^{-1} + \cdots + q^{-(m-1)}) \sum_{n=0}^{\infty} c_m(mn)q^{mn} \]

Proof. By Lemma 2.4,

\[ \sum_{n=0}^{\infty} c_m(mn)q^{mn} = 1 + \frac{q^m}{1-q^m} C_m(q^m). \]
On the other hand,

\[
C_m(q) = 1 + \frac{q}{1 - q} + \sum_{n=1}^{\infty} \frac{q^{m+\cdots m^n}}{(1 - q^m) \cdots (1 - q^{mn})} \cdot \frac{q}{1 - q}
\]

\[
= \frac{1}{1 - q} + \frac{q}{1 - q} \sum_{n=0}^{\infty} \frac{q^{m(1+\cdots m^n)}}{(1 - q^m) \cdots (1 - q^{m\cdot m^n})}
\]

\[
= \frac{1}{1 - q} + \frac{q}{1 - q} C_m(q^m).
\]

Therefore,

\[
C_m(q^m) = q^{-1}(C_m(q)(1 - q) - 1)
\]

and so

\[
\sum_{n=0}^{\infty} c_m(mn)q^{mn} = 1 + \frac{q^{m-1}}{1 - q^m}(C_m(q)(1 - q) - 1).
\]

Solving for \(C_m(q)\) gives the desired result.

Lemma 2.6 can now be used to prove that the values of the function \(c_m(n)\) come in \(m\)-tuples as described in the next lemma.

**Lemma 2.7.** For all \(n \geq 1\),

\[
c_m(mn) = c_m(mn - 1) = c_m(mn - 2) = \cdots = c_m(mn - (m - 1)).
\]

**Proof.** Compare coefficients of \(q^n\) on both sides of the identity in Lemma 2.6.

We now begin the consideration of \(c_m(mn)\) modulo \(m\) by proving the following lemma:

**Lemma 2.8.** If \(n \equiv k \pmod{m}\) where \(1 \leq k \leq m\), then

\[
c_m(mn) \equiv 1 + (k - 1)c_m(n) \pmod{m}.
\]

**Proof.** By Lemma 2.5,

\[
c_m(mn) = c_m(0) + c_m(1) + \cdots + c_m(n - 1).
\]

Next, we write \(n = jm + k\) for some integer \(j\). Then

\[
c_m(mn) = c_m(0) + c_m(1) + \cdots + c_m(m)
\]

\[
+ c_m(m + 1) + \cdots + c_m(2m)
\]

\[
+ \cdots + c_m((j - 1)m + 1) + \cdots + c_m((j - 1)m + m)
\]

\[
+ c_m(jm + 1) + \cdots + c_m(jm + k - 1)
\]

\[
\equiv 1 + c_m(jm + 1) + \cdots + c_m(jm + k - 1) \pmod{m}
\]

by Lemma 2.7

\[
\equiv 1 + (k - 1)c_m(jm + k) \pmod{m}
\]

by Lemma 2.7

\[
= 1 + (k - 1)c_m(n).
\]
Next, we prove an additional lemma involving an “internal” congruence satisfied by $c_m$ modulo $m$. It is interesting to note that a similar result holds for $b_m(n)$, the unrestricted $m$-ary partition function studied in [3, 5, 6].

**Lemma 2.9.** For all $n \geq 0$,

$$c_m(m^3n) \equiv c_m(mn) \pmod{m}.$$  

*Proof.* By Lemma 2.8, we know

$$c_m(m^3n) = c_m(m(2n))$$

$$\equiv 1 + (m - 1)c_m(m^2n) \pmod{m}$$

$$= 1 + (m - 1)c_m(m(mn))$$

$$\equiv 1 + (m - 1)(1 + (m - 1)c_m(mn)) \pmod{m}$$

$$\equiv c_m(mn) \pmod{m}.$$  

Lemma 2.9 enables a significant reduction in the number of cases which will need to be checked when we prove Theorem 2.1. This is because of the following. Given $n$ written in $m$-ary notation as

$$n = \alpha m^j + \beta m^k + \cdots + \gamma m^r,$$

we see immediately that

$$mn = \alpha m^{j+1} + \beta m^{k+1} + \cdots + \gamma m^{r+1},$$

where $\alpha, \beta, \ldots, \gamma \in \{1, 2, \ldots, m - 1\}$ and $j < k < \cdots < r$. Thus, we can divide by $m^2$ for as many times as we wish if $j \geq 2$ (because $j + 1 \geq 3$). Therefore, we only need to consider the cases $j = 0$ and $j = 1$ in what follows.

We are now in a position to prove Theorem 2.1 which provides a characterization of $c_m(mn)$ modulo $m$ simply based on the $m$-ary representation of $n$.

*Proof.* By Lemma 2.9, we see that if $j \geq 2$, then $m^3 | mn$. This means $c_m(mn) = c_m(\binom{n}{m})$. Thus, we may assume $j = 0$ or $j = 1$ without loss of generality.

Now we consider two cases (based on the parity of $j$).

- **Case 1:** $j$ is even, so we can assume $j = 0$. Hence,

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)c_m(n) \pmod{m}$$

$$= 1 + (\alpha_0 - 1)c_m(\alpha_0 + \alpha_1 m + \alpha_2 m^2 + \ldots).$$

Now since $m > \alpha_0 \geq 1$, we may replace $\alpha_0$ by $m$ (thanks to Lemma 2.7). Then the above becomes

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)c_m((\alpha_1 + 1)m + \alpha_2 m^2 + \ldots) \pmod{m}$$

$$= 1 + (\alpha_0 - 1)c_m(m((\alpha_1 + 1) + \alpha_2 m + \alpha_3 m^2 + \ldots))$$

$$\equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m((\alpha_1 + 1) + \alpha_2 m + \alpha_3 m^2 + \ldots)) \pmod{m}.$$ 

Now $1 \leq \alpha_1 + 1 \leq m$, so by Lemma 2.7 we may replace $\alpha_1 + 1$ by $m$ in the above to obtain

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m(m(\alpha_2 + 1) + \alpha_3 m + \ldots)) \pmod{m}.$$  

- **Case 2:** $j$ is odd, so we can assume $j = 1$. Hence,

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)c_m(\alpha_0 m + \alpha_1 m^2 + \ldots) \pmod{m}$$

$$= 1 + (\alpha_0 - 1)c_m(\alpha_1 m^2 + \alpha_2 m^3 + \ldots)$$

$$\equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m(\alpha_1 m^2 + \alpha_2 m^3 + \ldots)) \pmod{m}.$$ 

Now $0 \leq \alpha_1 + 1 \leq m$, so by Lemma 2.7 we may replace $\alpha_1 + 1$ by $m$ in the above to obtain

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m(m(\alpha_2 + 1) + \alpha_3 m + \ldots)) \pmod{m}.$$  

\[\square\]
Now $1 \leq \alpha_2 + 1 \leq m$, so we may apply Lemma 2.7 again, and the process continues until we hit some $\alpha_i = 0$ at which time the process terminates. The result is

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)(1 + \alpha_1(1 + \alpha_2(1 + \alpha_3 + \ldots))) \pmod{m}$$

$$= \alpha_0 + (\alpha_0 - 1) \sum_{i=1}^{\infty} \alpha_1 \alpha_2 \ldots \alpha_i$$

which is equivalent to the first case of Theorem 2.1.

- Case 2: $j$ is odd, so we can assume $j = 1$. Hence, $n \equiv m \pmod{m}$, and by Lemma 2.8,

$$c_m(mn) \equiv 1 - c_m(n) \pmod{m}$$

$$= 1 - c_m\left(m \sum_{j=0}^{\infty} \alpha_{j+1}m^j\right).$$

Now Case 1 above is applicable to $n' = \sum_{j=0}^{\infty} \alpha_{j+1}m^j$ because $1 \leq \alpha_1 < m$. Hence, the desired result follows.

With the goal of demonstrating the applicability of Theorem 2.1, we compute a few examples.

- Let $m = 4$, $n = 123 = 3 + 2 \cdot 4 + 3 \cdot 4^2 + 1 \cdot 4^3$. Then

$$c_4(4 \cdot 123) = c_4(492) = 5843 \equiv 3 \pmod{4}.$$  

This is an example of the case $j = 0$. Theorem 2.1 asserts that

$$c_4(4 \cdot 123) \equiv 3 + (3 - 1)(2 + 2 \cdot 3 + 2 \cdot 3 \cdot 1) \pmod{4}$$

$$= 3 + 2 \cdot 14$$

$$= 3 \pmod{4}$$

as computed above.

- Let $m = 5$, $n = 485 = 2 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3$. Then

$$c_5(5 \cdot 485) = c_5(2425) = 230358 \equiv 3 \pmod{5}.$$  

This is an example of the case $j = 1$. Theorem 2.1 asserts that

$$c_5(5 \cdot 485) \equiv 1 - 2 - (2 - 1)(4 + 4 \cdot 3) \pmod{5}$$

$$= 1 - 2 - 16$$

$$= -17$$

$$\equiv 3 \pmod{5}$$

as computed above.

REFERENCES


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