Adjoining to Wythoff's Game its P-positions as Moves

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Abstract. We adjoin to the generalized *Wythoff* Game three subsets of its P-positions as moves, resulting in three different classes of games. We analyze these classes, characterizing the P-positions of some and exhibiting equivalences between others.

KEYWORDS. Combinatorial games, Extended Wythoff games

1. Introduction

This paper is about a general way of producing a new game Γ_{i+1} from a known game Γ_i — applied to a particular game.

By *game* we mean a combinatorial game, i.e., two-player, perfect information (no hidden information such as in certain card games) without chance moves (no dice) and outcome restricted to (lose, win), (tie, tie) and (draw, draw) for the two players who play alternately: Nim (see below) and chess are examples. A *tie* is an end position in which neither player won, and a draw is a non-end position from which a player has a next nonlosing move, but cannot force a win. There are several ways of defining a win-position. Here are two: in *normal* play, the player making the last move wins and the opponent loses. This outcome is reversed for *misère* play.

We restrict attention to *classical impartial* games, i.e., those without ties or draws, where the options (moves) are the same for both players at every position; and to normal play. The theory of such games can be found in Conway [Con1976] and Berlekamp, Conway and Guy [BCG1982]. Background for this paper can be found in [Fra1996]. We denote by $\mathbb{Z}^0$ and $\mathbb{Z}^+$ the set of nonnegative integers and the set of positive integers respectively.

We now describe two particular games.

1. Nim. Given a finite number of piles, each containing finitely many tokens. A move consists of selecting a (single, nonempty) pile and removing from it a positive number of tokens, possibly the entire pile.

2. a-Wythoff's Game. Let $a \in \mathbb{Z}^+$ be a fixed parameter. Given two piles, each containing finitely many tokens. A move is either as in Nim, or else, tokens are removed from both piles, namely say k (>0) from one and l (>0) from the other, subject to the constraint $|k-l| < a$.

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REMARKS. (i). We consider both games in normal play, i.e., the player making the last move wins.

(ii). 1-Wythoff's game is known simply as *Wythoff's game* [Wyt1907]. It has also been analyzed in [Cox1953], [YaY1967]; the generalization is analyzed in [Fra1982].

The bare minimal tool required for analyzing games such as the above is the notion of P- and N-positions. Informally, a P-position is a position from which the Previous (2nd) player can force a win; and an N-position is one from which the Next (1st) player can force a win. Denote the set of all P-positions of a game by $\mathcal{P}$ and the set of all its N-positions by $\mathcal{N}$. For a position u of a game denote by $F(u)$ the set of all its *followers*, or *options*, i.e., all positions attainable by a single move from u. Then the following relationship holds.

$$u \in \mathcal{P} \text{ if and only if } F(u) \subseteq \mathcal{N}$$
$$u \in \mathcal{N} \text{ if and only if } F(u) \cap \mathcal{P} \notin \emptyset.$$

It is also a fact that the set of all positions of every game can be partitioned uniquely into its subsets $\mathcal{P}$ and $\mathcal{N}$.

We see that the P- and N-positions are not "symmetrical". A position in $\mathcal{P}$ requires that *all* its followers are in $\mathcal{N}$, which is a relatively rare event. Indeed, Singmaster has shown that "almost all" positions are N-positions [Sin1981], [Sin1982]. This fact may partly explain why a winning strategy of a game is normally given by characterizing its P-positions rather than its N-positions.

What are the P-positions of Nim and a-Wythoff's game? Denote the m pile sizes of a game of Nim by $k_1, \ldots, k_m$. Then the position $(k_1, \ldots, k_m)$ is in $\mathcal{P}$ if and only if $k_1 \oplus \ldots \oplus k_m = 0$, where $a \oplus b$ denotes the *Nim sum* (also known as XOR or addition over $GF(2)$) of a and b. In particular, for $m = 2$, the P-positions are those for which the two pile sizes are equal.

Let S be any finite subset of $\mathbb{Z}^0$, $\bar{S}$ its complement with respect to $\mathbb{Z}^0$. Let $\text{mex } S = \min \bar{S} =$ least nonnegative integer not in S. Note that the mex of the empty set is 0. Denote by $\{(A'_n, B'_n)\}_{n=0}^{\infty}$ the set of P-positions of a-Wythoff's game, where A'_n and B'_n denote the number of tokens in the two piles. Then for all $n \in \mathbb{Z}^0$,

$$A'_n = \text{mex}\{A'_m, B'_m : m < n\}, \; B'_n = A'_n + an.$$

Thus, for example, $(A'_0, B'_0) = (0, 0)$, $(A'_1, B'_1) = (1, a + 1)$.

It has been pointed out in §6 of [Fra1996] that interesting games can be obtained by adjoining to a given game an appropriate subset of its P-positions as moves. For example, 1-Wythoff's game is 2-pile Nim, to which Nim's P-positions have been adjoined as moves. This observation also enabled us to find the long-elusive "correct" generalization of 1-Wythoff's game to more than two piles, as pointed out there. The idea has also been exploited in [FrL1991] to examine games which bridge Nim and 1-Wythoff. Note that "adding a P-position as a move" means that this move is available to either player, since all our games are impartial.

In this paper we analyze three games, $\Gamma_1, \Gamma_2, \Gamma_3$, obtained from a-Wythoff's game by adjoining to it subsets of its P-positions as moves.

We let Γ_1 be the game obtained from a-Wythoff's game by adjoining to a-Wythoff's game the P-position $(A'_1, B'_1) = (1, a+1)$ as an additional move. Thus Γ_1 can be viewed as a-Wythoff's game, in which the condition $|k - l| < a$ has been relaxed by permitting to take also $k = a+1$ and $l = 1$ (with $k - l = a$) from the two piles.

By Γ_2 we denote the game obtained from a-Wythoff's game by adjoining to a-Wythoff's game the P-positions (A'_1, B'_1) and (A'_{a+3}, B'_{a+3}) as additional moves. In Γ_2 the condition $|k - l| < a$ has been relaxed further, which now has two exceptions.

Finally, by Γ_3 we denote the game obtained from a-Wythoff's game by adjoining to a-Wythoff's game *all* the nonzero P-positions $\cup_{i=1}^{\infty}(A'_i, B'_i)$ as additional moves. Thus in Γ_3, the condition $|k - l| < a$ has been broken infinitely often.

Since a-Wythoff's game depends on a parameter a, which can be any positive integer, so do the Γ_i. Thus each of them is actually an infinite class of games. But we may think of a as fixed, when convenient, whence each Γ_i is a single game.

In §2 we prove some preliminary results, useful throughout the paper. The main results are enunciated in the four theorems. In Theorem 1 of §3 we give a complete characterization of the P-positions of Γ_1 for all $a > 1$.

Definition 1 . *Two games with the same set of positions but possibly different move-rules are* equivalent *if their P-positions are the same.*

Note that two equivalent games have, additionally, the same set of N-positions, and so also the same winning strategy.

In Theorem 2 of §4 we prove that for $a = 2$, Γ_1 and Γ_3 are equivalent. Useful properties of the P-positions of Γ_2 for all $a > 2$ are given in Theorem 3 of §5. They enabled us to compute the first 100,000 P-positions of Γ_2 within a second on a computer. Though the data reveals interesting relationships, the observed regularities seem to get broken after a while and replaced by new ones. We didn't succeed in characterizing these P-positions. The situation is reminiscent to that of many games, such as Grundy's game [BCG1982, Ch. 4].

In Theorem 4 of the final §6 we prove that for all $a > 2$, Γ_2 and Γ_3 are equivalent.

2. Preliminary Results

Notations.

1. We consider the following moves for our games.

Type I. Remove any positive number of tokens from a single nonempty pile (Nim move).

Type II. Remove tokens from both piles, say k (> 0) from one and l (> 0) from the other, subject to the constraint $|k - l| < a$ (a-Wythoff move, together with Type I moves).

Type III. Remove A'_n from one pile and B'_n from the other for some $n > 0$ (moves for Γ_i, together with Type I and II moves).

2. The set $\{(A'_j, B'_j)\}_{j=0}^{\infty}$ denotes the P-positions of a-Wythoff's game.

3. We denote by $\{(A_n, B_n)\}_{n=0}^{\infty}$ the set of P-positions of all our games, other than a-Wythoff's game, such that for every $n \in \mathbb{Z}^0$, $A_n \le A_{n+1}$ and $A_n \le B_n$. In general, game positions are denoted by (x, y) with $x \le y$, where x and y denote the number of tokens in the two piles.

4. $A = \cup_{i=1}^{\infty} A_i$ and $B = \cup_{i=1}^{\infty} B_i$.

5. $\lfloor x \rfloor$ denotes the largest integer $\le x$.

Lemma 1 . *For all* $j \in \mathbb{Z}^0$, $A'_j \ge j$.

Proof. By [Fra1982], $A'_j = \lfloor j\alpha \rfloor$ where $\alpha = (2 - a + \sqrt{a^2 + 4})/2$. In order to prove that $A'_j = \lfloor j\alpha \rfloor \ge j$, it suffices to show that $\alpha \ge 1$. Indeed, $\alpha > (2 - a + a)/2 = 1$. ■

Lemma 2 . *For all* $a > 2$, $(A'_{a+3}, B'_{a+3}) = (a + 4, (a + 2)^2)$.

Proof. From [Fra1982], $A'_m = \text{mex}\{A'_i, B'_i : 0 \le i < m\}$ and $B'_m = A'_m + ma$ for all $m \ge 0$. Thus, $A'_0 = \text{mex}\{\emptyset\} = 0$ and so $B'_0 = 0$. Also $A'_1 = \text{mex}\{0\} = 1$ and $B'_1 = A'_1 + a = 1 + a$. Then $A'_j = j$ and $B'_j = j(a + 1)$ for all $j \le a$. Since $a > 2$ implies $a + 4 < 2a + 2 = B'_2$, we have $A'_{a+i} = a + i + 1$ for $i \in \{1, 2, 3\}$, so $B'_{a+3} = a + 4 + (a + 3)a = (a + 2)^2$. ■

Lemma 3 . *Let* $\{(A_j, B_j)\}_{j=0}^{\infty}$ *be the* P*-positions of* Γ_i *for any* $i \in \{1, 2, 3\}$. *Then* (i) $B_n > A_n > A_m$ *for every* $n > m \ge 0$, *and* (ii) $A \cap B = \emptyset$.

Proof. (i) Let $n > m \ge 0$. Suppose that $A_m = A_n$. Now, $B_m \ne B_n$ (otherwise $(A_m, B_m) = (A_n, B_n)$). If $B_n > B_m$ then $(A_n, B_n) \to (A_m, B_m)$ is a legal move (of Type I), which contradicts the fact that (A_n, B_n) is a P-position of Γ_i. Similarly, if $B_n < B_m$, then $(A_m, B_m) \to (A_n, B_n)$ is a legal move, resulting in the same contradiction. Thus $A_m \ne A_n$, so by Notation 2, $A_n > A_m$.

Again by Notation 2, $A_n \le B_n$. If $A_n = B_n$ for some $n > 0$, then the move $(A_n, B_n) \to (0, 0)$ is legal since $|(B_n - 0) - (A_n - 0)| = |B_n - A_n| = 0 < a$ (a move of Type II). Therefore we get $B_n > A_n > A_m$.

(ii) Suppose $A_n = B_m$ for some $m, n \in \mathbb{Z}^+$. Then $m \neq n$ by (i), and $B_n > A_n = B_m > A_m$. It follows that $(A_n, B_n) \to (A_m, B_m)$ is a legal move, ($B_n \to A_m$; a move of Type I), a contradiction. Thus $A \cap B = \emptyset$. ∎

Lemma 4 . *Let $\{(A_j, B_j)\}_{j=0}^{\infty}$ be the P-positions of Γ_i for any $i \in \{1, 2, 3\}$. Then $A \cup B = \mathbb{Z}^+$.*

Proof. Clearly $A \cup B \subseteq \mathbb{Z}^+$. Suppose there exists $s \in \mathbb{Z}^+$ with $s \notin A \cup B$. Then for every $t \in \mathbb{Z}^0$, $(s, t) \notin \cup_{i=0}^{\infty}(A_i, B_i)$. Hence (s, t) is an N-position for every $t \in \mathbb{Z}^0$. We show below that there is some $t_0 \in \mathbb{Z}^0$ for which (s, t_0) is not an N-position, which is clearly a contradiction. It suffices to find such t_0 for which no follower of (s, t_0) is a P-position. First, the number of P-positions to which we may move from (s, t), $t \in \mathbb{Z}^0$ is bounded. Indeed, let $n_0 = \max\{i : A_i < s\}$. Then from (s, t) we can only move to (A_i, B_i) for some $i \leq n_0$, whatever t is. Secondly, we show that the number of possible moves of Type III from (s, t) is also bounded. It is certainly bounded by the *index* $m_0 = j$ of the largest Type III move (A'_j, B'_j) from (s, t), which is:

$$m_0 \leq \begin{cases} 1 & \text{for } \Gamma_1 \\ a + 3 & \text{for } \Gamma_2 \\ \max\{i : A'_i \leq s\} & \text{for } \Gamma_3. \end{cases}$$

For Γ_1 and Γ_2 this follows from the definition of these games. For Γ_3 it follows from the fact that $\{A'_j\}_{j=0}^{\infty}$ is a monotonically increasing sequence and $B_j \geq A_j$ for all $j \geq 0$ (see [Fra1982]), so moves (A'_i, B'_i) from (s, t) can be made only for $i \leq m_0$.

Now let $D = \max\{B_i : i \leq n_0\} + 1$, and $t_0 = D + s(a + 1)$. We show that from position (s, t_0), none of the three types of moves leads to a P-position.

(i) $t_0 > \max\{B_i : i \leq n_0\} \geq B_i > A_i$ for every $i \leq n_0$. Hence $t_0 \notin \cup_{i=0}^{n_0} A_i \cup_{i=0}^{n_0} B_i$. Since also $s \notin \cup_{i=0}^{n_0} A_i \cup_{i=0}^{n_0} B_i$, no move of Type I from (s, t_0) can lead to a P-position.

(ii) For $i \leq n_0$, $|(D + s(a + 1) - B_i) - (s - A_i)| = |(D - B_i) + A_i + sa| > A_i + sa \geq a$. Therefore no move of Type II results in a P-position.

(iii) According to Lemma 1, for all $j \in \mathbb{Z}^0$, $j \leq A'_j$. Therefore $B'_{m_0} = A'_{m_0} + m_0 a \leq A'_{m_0} + A'_{m_0} a \leq s(a + 1)$ since $s \geq A'_{m_0}$. (If $s < A'_{m_0} < B'_{m_0}$ then (A'_{m_0}, B'_{m_0}) is not a legal move.)

The maximum Type III move from (s, t_0) is (A'_{m_0}, B'_{m_0}). But $t_0 - B'_{m_0} = D + s(a + 1) - B'_{m_0} \geq D > B_i > A_i$ for every $i \leq n_0$. Hence we cannot move from (s, t_0) to any (A_i, B_i), $i \leq n_0$ by a move of Type III.

In conclusion, we proved that there is no $i \in \mathbb{Z}^0$ such that the move $(s, t_0) \to (A_i, B_i)$ is legal. Therefore $s \in A \cup B$. ∎

DEFINITION 2 . *Two subsets A and B of positive integers are* complementary *if $A \cap B = \emptyset$ and $A \cup B = \mathbb{Z}^+$.*

COROLLARY 1 . *For each of $\Gamma_1, \Gamma_2, \Gamma_3$, the sets A and B are complementary.*

PROOF. Follows directly from Lemmas 1 and 3. ■

COROLLARY 2 . *For each of $\Gamma_1, \Gamma_2, \Gamma_3$, and every $n \in \mathbb{Z}^0$ we have, $A_n =$ $\text{mex}\{A_i, B_i : 0 \le i < n\}$.*

PROOF. Clear for $n = 0$. For $n > 0$, let $t = \text{mex}\{A_i, B_i : 0 \le i < n\}$. Then $t > 0$. If $A_n < t$ then $A_n = A_i$ for some $i < n$ or $A_n = B_i$ for some $i < n$, contradicting Lemma 3. Hence $A_n \ge t$. Now $\{A_i\}_{i=0}^{\infty}$ is an increasing sequence and $B_i > A_i$ for all $i > 0$. Thus if $A_n > t$, then $t \notin A \cup B$, contradicting Lemma 4. Hence $A_n = t$. ■

3. The P-Positions of Γ_1

DEFINITION 3 . *For every $n \in \mathbb{Z}^+$, let*

$$G_n = \left\lfloor \tfrac{(a+2)n-1}{a+1} \right\rfloor, \; H_n = (a+2)n.$$

We also put $G_0 = H_0 = 0$.

LEMMA 5 . *The sets $G = \cup_{i=1}^{\infty} G_i$ and $H = \cup_{i=1}^{\infty} H_i$ are complementary.*

PROOF. Follows directly from Theorem III in [Fra1969]. ■

Dividing G_n by $a+2$, we get a quotient q and remainder $r \in \{1, \dots, a+2\}$. However, since G and H are complementary, $r \neq a+2$.

LEMMA 6 . *For any $n \in \mathbb{Z}^+$, write $G_n = (a+2)q + r$, $r \in \{1, \dots, a+1\}$. Then $n = (a+1)q + r$.*

PROOF. For s/t rational, we use $(s+1)/t - 1 \le \lfloor s/t \rfloor \le s/t$. Since $G_n = \lfloor ((a+2)n-1)/(a+1) \rfloor = (a+2)q+r$, we get $(a+2)q+r \le ((a+2)n-1)/(a+1) \le (a+2)q + r + a/(a+1)$, which implies $n \le (a+1)q + r + (a+1-r)/(a+2) < (a+1)q + r + 1$ and $n \ge (a+1)q + r - (r-1)/(a+2) > (a+1)q + r - 1$. Since n is an integer, we conclude $n = (a+1)q + r$. ■

LEMMA 7 . *Let $n \in \mathbb{Z}^+$. For any $i \in \{0, \dots, n-1\}$, let $k = G_n - G_i$ and $l = H_n - H_i$. Then $l - k \ge a$.*

PROOF. If $i = 0$ then $l - k = H_n - G_n = (a+2)n - \lfloor((a+2)n-1)/(a+1)\rfloor \geq (a(a+2)n+1)/(a+1) = an + (an+1)/(a+1) \geq a$, since $n \geq 1$. If $i > 0$, then $l - k = (a+2)(n-i) - \lfloor((a+2)n-1)/(a+1)\rfloor + \lfloor((a+2)i-1)/(a+1)\rfloor \geq (a+2)(n-i) - ((a+2)n-1)/(a+1) + ((a+2)i-1)/(a+1) - 1 + 1/(a+1) = (a(a+2)(n-i)-a)/(a+1) = a(n-i) + a(n-i-1)/(a+1) \geq a$. ■

THEOREM 1 . *For $a > 1$, let $\cup_{n=0}^{\infty}\{(A_n, B_n)\}$ be the P-positions of Γ_1. Then $A_n = G_n$, $B_n = H_n$ for all $n \geq 0$.*

PROOF. It suffices to show:
I. No position (G_n, H_n) has a follower of the form (G_i, H_i).
II. Given any position $(x, y) \neq (G_m, H_m)$, for all $m \in \mathbb{Z}^0$, there is a move to (G_n, H_n) for some $n \in \mathbb{Z}^0$.

Proof of I. Since the sets G and H are complementary (Lemma 5), the move $(G_n, H_n) \to (G_i, H_i)$ cannot be a move of Type I. The sequences $\{G_i\}_{i=1}^{\infty}$ and $\{H_i\}_{i=1}^{\infty}$ are evidently increasing. Hence we have to consider only potential followers (G_i, H_i) for $i < n$.

Can a move $G_n \to G_i$, $H_n \to H_i$ be a Type II move for some i? Lemma 7 shows that the answer is negative. Moreover, since $H_n - H_i = (a+2)(n-i) \geq a+2 > a+1$, this move can't be a Type III move either.

Now, we need also to prove that it's impossible to move $G_n \to H_i$ and $H_n \to G_i$. Let $k_2 = G_n - H_i$ $(G_n > H_i)$, $l_2 = H_n - G_i$, $n > i > 0$ (we already took care of the case where $i = 0$ since $G_0 = H_0 = 0$).

First consider a move of Type II.

$$\begin{aligned} |l_2 - k_2| &= |(H_n - G_i) - (G_n - H_i)| = |(H_n - G_n) + (H_i - G_i)| \\ &= (H_n - G_n) + (H_i - G_i) = (H_n - H_i) - (G_n - G_i) + 2(H_i - G_i) \\ &= |l - k| + 2(H_i - G_i) \geq |l - k| \geq a, \end{aligned}$$

where l and k are as in Lemma 7, contradicting $|l_2 - k_2| < a$.

Secondly, consider a move of Type III. Now, $n > 1$ since $i > 0$. But $H_n - G_i \geq H_n - G_n \geq an$ (proof of Lemma 2) $\geq 2a > a+1$. Therefore also this option doesn't exist, ending the proof of I.

Proof of II. Let (x, y), with $x \leq y$, be a position not of the form (G_m, H_m), for all $m \in \mathbb{Z}^0$. We may also assume that $x > 0$. Since G and H are complementary sets, we have $x \in G$ or $x \in H$; and also $G_{n+1} - G_n \in \{1, 2\}$ for all $n \geq 0$.

Case (i). $x = H_n$ for some $n > 0$. Then move $y \to G_n$.

Case (ii). $x = G_n$ for some $n \geq 0$. We consider four possibilities, three of which are very simple.
1. $y > H_n$. Then move $y \to H_n$.
2. $y = H_n - 1$. Then move $y \to H_{n-1}$, $x \to G_{n-1}$. This is a legal move since we

take either $(2, a+1)$ (move of Type II; if $G_n - G_{n-1} = 2$), or $(1, a+1)$ (move of Type III; if $G_n - G_{n-1} = 1$).
3. $y = G_n + k$, where $k \in \{0, \ldots, a-1\}$. Then move $y \to 0$, $x \to 0$. This is also a legal move since $|y - x| < a$.
4. $G_n + a \le y \le H_n - 2$. Put $d = y - x = y - G_n$. Then $a \le d \le H_n - G_n - 2$. Let

$$m = \begin{cases} 1 & \text{if } d = a \\ \lfloor \frac{(a+1)d+a-1}{a(a+2)} \rfloor & \text{if } a < d \le H_n - G_n - 2. \end{cases}$$

Then the move $(x, y) \to (G_m, H_m)$ is legal. In fact, we show: (i) $m < n$, (ii) $y \ge H_m$, and (iii) $|(y - H_m) - (x - G_m)| < a$.

(i) $a \le d \le H_n - G_n - 2 = (a+2)n - \left\lfloor \dfrac{(a+2)n - 1}{a+1} \right\rfloor - 2 \le \dfrac{a(a+2)n}{a+1} - 1.$

Hence $\quad a \le \dfrac{a(a+2)n}{a+1} - 1, \quad$ so $n \ge \dfrac{(a+1)^2}{a(a+2)} > 1.$
Thus, for $d = a$, we have $1 = m < n$. Now, m is a nondecreasing function of d, so the maximum value of m, for every given n, is reached at the maximum value of d.

$d_{max} = H_n - G_n - 2 \le \frac{a(a+2)n}{a+1} - 1$. Therefore, $\; m \le m(d_{max}) \le$

$$\left\lfloor \frac{(a+1)(\frac{a(a+2)n}{a+1} - 1) + a - 1}{a(a+2)} \right\rfloor = \left\lfloor n - \frac{2}{a(a+2)} \right\rfloor = n - 1 < n.$$

(ii) By (i), $n > m \ge 1$. For fixed d, let $n_0 = \min\{n : G_n + d \le H_n - 2\}$. There clearly exists such $n_0 \in \mathbb{Z}^+$. Then $G_{n_0-1} + d > H_{n_0-1} - 2$, so $G_{n_0-1} + d \ge H_{n_0-1} - 1$. Now $G_{n_0-1} + 1 \le G_{n_0}$. It follows that $y = G_n + d \ge G_{n_0} + d \ge G_{n_0-1} + d + 1 \ge H_{n_0-1} \ge H_m$. The last inequality is due to the fact that $n_0 > m$, by (i).

(iii) $|(y - H_m) - (x - G_m)| = |(y - x) - (H_m - G_m)| = |d - (H_m - G_m)|$. For $d = a$: $|a - (H_1 - G_1)| = |a - ((a+2) - 1)| = 1 < a$. For $d > a$: $-H_m + G_m = -(a+2)m + \left\lfloor \dfrac{(a+2)m - 1}{a+1} \right\rfloor = \left\lfloor \dfrac{-(a(a+2)m + 1)}{a+1} \right\rfloor$.
Now, $\dfrac{(a+1)(d-a)}{a(a+2)} \le m \le \dfrac{(a+1)d + a - 1}{a(a+2)}$. It follows that:

$$-H_m + G_m \ge \left\lfloor \frac{-a(a+2)\frac{(a+1)d+a-1}{a(a+2)} - 1}{a+1} \right\rfloor = \left\lfloor -d - \frac{a}{a+1} \right\rfloor = -d - 1,$$

$$-H_m + G_m \le \left\lfloor \frac{-a(a+2)\frac{(a+1)(d-a)}{a(a+2)} - 1}{a+1} \right\rfloor = \left\lfloor -(d-a) - \frac{1}{a+1} \right\rfloor =$$

$-(d-a)-1 = -d+(a-1)$.

The last two inequalities imply $-(a-1) \le -1 \le d-(H_m-G_m) \le a-1$, which implies $|d-(H_m-G_m)| \le a-1 < a$. ■

4. An Equivalence for $a = 2$

THEOREM 2 . *For $a = 2$, Γ_1 and Γ_3 are equivalent.*

PROOF. The difference in move-rules of the two games is that Γ_3 has the infinity $\cup_{i=1}^{\infty}(A'_i, B'_i)$ of Type III moves, whereas Γ_1 has only a single Type III move, namely (A'_1, B'_1). By Theorem 1 it suffices to show that, for every $n \in \mathbb{Z}^+$, no move (A'_i, B'_i) from (G_n, H_n) leads to any (G_i, H_i). Our proof method is similar to that of Theorem 1, part I. We consider two cases.

(i) $i = 0$. We need to prove that, for every $n \in \mathbb{Z}^+$, the move $(G_n, H_n) \to (0,0)$ is not a move of Type III, i.e., $(G_n, H_n) \ne (A'_j, B'_j)$ for every $j \in \mathbb{Z}^+$. If we write $G_n = 4q+r$, $r \in \{1,2,3\}$, then by Lemma 6, $n = 3q+r$, so $H_n = 12q+4r$. Suppose now that $(G_n, H_n) = (A'_j, B'_j)$ for some j. Then $2j = B'_j - A'_j = H_n - G_n = 8q+3r$. Since j is an integer, $r = 2$, so $j = 4q+3$. It follows that $A'_j = G_n = 4q+2 = j-1 < j$. However, according to Lemma 1, $A'_j \ge j$ for all $j \ge 0$, which rejects this case.

(ii) $i > 0$. Let $G_n = 4q_n + r_n$, $G_i = 4q_i + r_i$. By Lemma 6, $n = 3q_n + r_n$, $i = 3q_i + r_i$, $r_n, r_i \in \{1,2,3\}$. Suppose now that $(G_n - G_i, H_n - H_i) = (A'_j, B'_j)$. Then $A'_j = 4(q_n - q_i) + (r_n - r_i)$ and $B'_j = 12(q_n - q_i) + 4(r_n - r_i)$. Thus, $2j = B'_j - A'_j = 8(q_n - q_i) + 3(r_n - r_i)$.

There are now three possibilities for the remainders r_n and r_i.

(a) $r_n = r_i$. Then $j = 4(q_n - q_i) = A'_j = \lfloor\sqrt{2}j\rfloor$. Thus $\sqrt{2}j - 1 < j \le \sqrt{2}j$, so $0 \le j < 1/(\sqrt{2}-1)$, which is true only for $j \in \{0,1,2\}$. However, $j = 4(q_n - q_i) \ge 4$ contradicts this.

(b) $r_n = 3$, $r_i = 1$. Then $A'_j = 4(q_n - q_i) + 2 = j - 1 < j$, contradicting Lemma 1.

(c) $r_n = 1$, $r_i = 3$. Then $j = 4(q_n - q_i) - 3$. Now $A'_j = 4(q_n - q_i) - 2 = j + 1 = \lfloor\sqrt{2}j\rfloor$. As in case (a), $\sqrt{2}j - 1 < j + 1 \le \sqrt{2}j$. It follows that $1/(\sqrt{2}-1) \le j < 2/(\sqrt{2}-1)$. Hence $j \in \{3,4\}$ which contradicts the fact that $j \equiv 1(\text{mod}4)$.

It remains only to prove that $G_n \to H_i$ and $H_n \to G_i$ with $i > 0$, isn't a possible Type III move. Let $k_2 = G_n - H_i$ $(G_n > H_i)$, and $l_2 = H_n - G_i$. Suppose $(k_2, l_2) = (A'_{j_2}, B'_{j_2})$ (a move pf Type III) for some j_2. Then $j_2 = (B'_{j_2} - A'_{j_2})/2 = ((H_n - G_n) + (H_i - G_i))/2$. Thus $A'_{j_2} = G_n - H_i = j_2 + (3G_n - H_n - 3H_i + G_i)/2$. By Definition 3, $H_n > 3G_n$ if $n > 0$ and $a = 2$. Therefore $A'_{j_2} < j_2 - (H_i - G_i)/2 - H_i < j_2$. This contradicts the fact that $A'_{j_2} \ge j_2$ for all $j_2 \geq 0$. ■

5. P-Positions of Γ_2 for $a > 2$

THEOREM 3 . *Let* $\cup_{n=0}^{\infty}\{(A_n, B_n)\}$ *with* $A_n \leq B_n$*, be the* P*-positions of* Γ_2 *with* $a > 2$*. Then for all* $n \in \mathbb{Z}^+$*,*
(1) $A_n - A_{n-1} \in \{1, 2\}$*,*
(2) $B_n - B_{n-1} \in \{a+2, a+3\}$*, and*
(3) $D_n - D_{n-1} \in \{a, a+1\}$*, where* $D_i = B_i - A_i$*.*

PROOF. By Lemma 2, Γ_2 is a-Wythoff's game to which the moves $(1, a+1)$ and $(a+4, (a+2)^2)$ have been adjoined.

We prove **(1)** and **(2)** simultaneously by induction on n. Base of the induction:

n	A_n	B_n
0	0	0
1	1	$a+2$.

Indeed, $A_1 = \text{mex}\{0\} = 1$ by Corollary 2. Thus, $A_1 - A_0 = 1$ and $B_1 - B_0 = a + 2$. Suppose that the two claims hold for all $n < m$ $(m > 0)$.

(1), **(2)** By Lemma 3, $A_m - A_{m-1} > 0$. Suppose $A_m - A_{m-1} > 2$. Then by Corollary 2, $A_{m-1} < A_{m-1}+1 < A_{m-1}+2 < A_m = \text{mex}\{A_k, B_k : 0 \leq k < m\}$. In particular, $A_{m-1} + 1, A_{m-1} + 2 \in \{A_k, B_k : 0 \leq k < m\}$. Since A and B are complementary (Corollary 1), it follows that $A_{m-1} + 1, A_{m-1} + 2 \in \cup_{k=0}^{m-1} B_k$, which contradicts the induction hypothesis for **(2)**. Thus at least one of $A_{m-1} + 1, A_{m-1} + 2$ is in A.

We note two facts.

(i) By Lemma 3, $B_i > A_i$ for all $i > 0$. Hence $D_i > 0$ for all $i > 0$.

(ii) We cannot move from a P-position to another P-position, e.g., by a move of Type II. Therefore $|(B_i - B_j) - (A_i - A_j)| = |D_i - D_j| \geq a$ for every $i \neq j$. In particular, $D_i \neq D_j$ for every $i \neq j$.

By (i), $D_m > 0 = D_0$. Suppose that $D_m < D_{m-1}$. Choose k such that $k = \min\{i : D_m < D_i\}$. Then $0 < k \leq m - 1$, and so $D_{k-1} < D_m < D_k$. By (ii), $D_m - D_{k-1} \geq a$ and $D_k - D_m \geq a$. Summing the two inequalities gives $D_k - D_{k-1} \geq 2a > a + 2$. But by the induction hypotheses for **(1)** and **(2)**, $D_k - D_{k-1} = B_k - B_{k-1} - (A_k - A_{k-1}) \leq a + 3 - 1 = a + 2$, a contradiction. Hence $D_m \geq D_{m-1}$. Moreover, (ii) implies $D_m \geq D_{m-1} + a$.

Now, $B_m = D_m + A_m \geq D_{m-1} + A_m + a$. Let

$$y_j = D_{m-1} + A_m + a + j,\ j \in \mathbb{Z}^0.$$

We prove below that $B_m \in \{y_0, y_1\}$ by showing that for precisely one $j \in \{0, 1\}$ and for all $i < m$, no permissible Type III move $M : (A_m, y_j) \to (A_i, B_i)$ is possible. First, M cannot be a move of Type I, since obviously $A_m > A_i, A_m \neq B_i$, and $y_j > B_i > A_i$. Furthermore, M cannot be a move of Type II, since: $|(y_j - B_i) - (A_m - A_i)| = |(y_j - A_m) - D_i| = |(D_{m-1} - D_i) + a + j| \geq a$. Therefore we are left only with the possibility that M is a move of Type III: $(A'_1, B'_1) = (1, a+1)$ or $(A'_{a+3}, B'_{a+3}) = (a+4, (a+2)^2)$. Note that

$$\begin{aligned} y_j - A_i \geq y_j - B_i &= A_m + D_{m-1} + a + j - B_i \\ &= A_m - A_i + D_{m-1} - D_i + a + j > A_m - A_i \geq A_m - B_i. \end{aligned}$$

Now $B'_k > A'_k$ for $k \geq 1$. Thus, since $y_j - A_i > A_m - B_i$ and also $y_j - B_i > A_m - A_i$, it follows that the only possible Type III move from M is to subtract B'_k from y_j and A'_k from A_m for $k \in \{1, a+3\}$. We consider two cases.

(a) $A_m = A_{m-1} + 1$. Then $y_j = D_{m-1} + A_{m-1} + a + j + 1 = B_{m-1} + a + j + 1$. Obviously $y_0 \neq B_m$, since $(A_m, y_0) \to (A_{m-1}, B_{m-1})$ using the move $(1, a+1)$. But using the move $(1, a+1)$ from (A_m, y_1), clearly leads to no (A_i, B_i), $i < m$. So it remains only to show that using the move $(a+4, (a+2)^2)$ on (A_m, y_1) leads to no (A_i, B_i), $i < m$. Let $x = A_m - (a+4) = A_{m-1} - (a+3)$. Since $A_m = \text{mex}\{A_i, B_i : 0 \leq i < m\}$, x is either some A_{j_1}, or B_{j_2}, $j_1, j_2 < m$.

(a.1) $x = A_{m-1} - (a+3) = A_{j_1}$.

$$\ldots \overbrace{A_{j_1}, \underbrace{\ldots}_{a+2}, A_{m-1}}^{a+4}, \ldots$$

There must be precisely one B_i $(i \leq m-1)$, such that $A_{j_1} < B_i < A_{m-1}$. For if there is none, then there is a gap of $(a+4)$ between some B_{k-1} and B_k, $k \leq m-1$, contradicting the induction hypothesis $a+2 \leq B_i - B_{i-1} \leq a+3$, for all $i \leq m-1$. If there were two, say B_{i-1} and B_i, then we would have $B_i - B_{i-1} \leq a+1$, contradicting the same induction hypothesis. Hence, $j_1 = m-1-(a+3)+1 = m-(a+3)$. Thus, $y_1 - (a+2)^2 = B_{m-1} + (a+2) - (a+2)^2 = B_{m-1} - (a+1)(a+2) \geq B_{m-1-(a+1)} > B_{m-(a+3)} = B_{j_1}$. It follows that there is no legal move leading from (y_1, A_m) to any other P-position. Therefore, in this case, $B_m = y_1$, so $B_m - B_{m-1} = a+2$.

(a.2) $x = A_{m-1} - (a+3) = B_{j_2}$.

$$\ldots B_{j_2}, \overbrace{\ldots, A_{m-1}}^{a+3}, \ldots$$

By considerations similar to those in **(a.1)**, the only possibility for this to happen is that $A_{m-1} - 1 = B_{j_2+1}$. Therefore, $A_{m-1-(a+1)} = A_{m-(a+2)} = B_{j_2} + 1 > B_{j_2}$. Then $y_1 - (a+2)^2 = B_{m-1} - (a+1)(a+2) \geq B_{m-(a+2)} > A_{m-(a+2)} > B_{j_2} \geq A_{j_2}$. Therefore $B_m = y_1$, so again $B_m - B_{m-1} = a+2$.

(b) $A_m = A_{m-1} + 2$. Then $y_j = D_{m-1} + A_{m-1} + 2 + a + j = B_{m-1} + (a+j+2)$. Now, $y_0 - (a+1) = B_{m-1} + 1 \geq A_{m-1} + 1 = A_m - 1$. Moreover, $A_{m-1} < A_m - 1 < A_m$, so $A_m - 1 \notin A$. Hence $(A_m - 1, y_0 - (a+1))$ is not a P-position, so the move $(1, a+1)$ doesn't lead to a P-position. It remains to check the move $(a+4, (a+2)^2)$ from (A_m, y_j). Let $x = A_m - (a+4) = A_{m-1} - (a+2)$. Now x is either some A_{j_1}, or B_{j_2}, $j_1, j_2 < m$.

(b.1) $x = A_{m-1} - (a+2) = A_{j_1}$.

$$\ldots \overbrace{A_{j_1}, \underbrace{\ldots}_{\text{a+1}}, A_{m-1}}^{a+3}, \ldots$$

As above, there is precisely one B_k, $k < m$, such that $A_{j_1} < B_k < A_{m-1}$. It follows that $j_1 = m-1-(a+2)+1 = m-(a+2)$. If $y_0 - (a+2)^2 = B_{m-1} + (a+2) - (a+1)^2 = B_{m-1} - (a+1)(a+2) \neq B_{m-1-(a+1)}$ then we are done and $B_m = y_0 = B_{m-1} + (a+2)$. Otherwise, if $y_0 - (a+2)^2 = B_{m-1-(a+1)}$, then $B_m = y_1 = B_{m-1} + (a+3)$. In the former case $B_m - B_{m-1} = a+2$, and in the latter $B_m - B_{m-1} = a+3$.

(b.2) $x = B_{j_2}$.

$$\ldots B_{j_2}, \overbrace{\ldots}^{\text{a+1}}, A_{m-1}, \ldots$$

Here $B_{j_2} < B_k < A_{m-1}$ holds for no $k \in \mathbb{Z}^0$. Thus, $A_{m-1-(a+1)} = B_{j_2} + 1 > B_{j_2}$. Now, $y_0 - (a+2)^2 = B_{m-1} - (a+1)(a+2) \geq B_{m-1-(a+1)} \geq A_{m-(a+2)} > B_{j_2} \geq A_{j_2}$. Thus, $B_m = y_0$, so $B_m - B_{m-1} = a+2$.

In conclusion, the induction hypothesis for **(2)** implies $A_m = A_{m-1} + 1$ or $A_m = A_{m-1} + 2$. The induction hypotheses for both **(1)** and **(2)** imply that in the former case, $B_m - B_{m-1} = a+2$; and in the latter case, $B_m - B_{m-1} = a+2$ or $B_m - B_{m-1} = a+3$, ending the proofs of **(1)** and **(2)**.

(3) Follows immediately from **(1)**, **(2)** and their proofs. Let k be any positive integer. We have, $D_k - D_{k-1} = (B_k - B_{k-1}) - (A_k - A_{k-1})$. If $A_k = A_{k-1} + 1$ then $D_k - D_{k-1} = a + 2 - 1 = a + 1$. Otherwise, $D_k - D_{k-1} = a$ or $a+1$. ■

COROLLARY 3 . *Let* $\cup_{n=0}^{\infty}\{(A_n, B_n)\}$ *with* $A_n \leq B_n$, *be the P-positions of* Γ_2 *with* $a > 2$. *If* $A_m = A_{m-1} + 1$, *then* $B_m - B_{m-1} = a + 2$. *If* $A_m = A_{m-1} + 2$ *then* $B_m - B_{m-1} = (a+2)$ *or* $(a+3)$.

PROOF. Follows directly from the proof of Theorem 3 **(2)**.

6. An Equivalence for $a > 2$

THEOREM 4 . *For all* $a > 2$, Γ_2 *is equivalent to* Γ_3.

PROOF. Γ_3 is obtained by adjoining to Γ_2 the moves $\cup_{i \notin \{0,1,a+3\}}(A'_i, B'_i)$. We show that the addition of these moves leaves the P-positions $\cup_{i=0}^{\infty}(A_i, B_i)$ of Γ_2 invariant. The proof is based on Theorem 3.

Suppose that $A'_k = A_n - A_i$ for some $n > i \geq 0$, $k > 0$. Clearly $n-i$ assumes its smallest value if between A_i and A_n there is a maximum number of elements of B, so a minimum distance of $a+2$ between consecutive B-elements.

$$\ldots A_i, \overbrace{\underbrace{B_{j_1}, A_{i_1}, \ldots, A_{i_{a+1}}}_{a+2}, \underbrace{B_{j_2}, A_{i_{a+2}}, \ldots, A_{i_{2a+2}}}_{a+2}, \ldots, A_n}^{A'_k}, \ldots$$

Dividing A'_k by $a+2$ gives, $A'_k = (a+2)q + r, \quad r \in \{0, \ldots, a+1\}$. Note that for $r \geq 2$, there may be an additional B-element between A_i and A_n. Thus,

$$n - i \geq (a+1)q + \begin{cases} r & r \in \{0,1\} \\ r-1 & r \in \{2, \ldots, a+1\}. \end{cases}$$

Now, assume that $B'_k = B_n - B_i$. Then $B'_k \geq (a+2)(n-i) \geq$

$$(a+1)(a+2)q + \begin{cases} (a+2)r & r \in \{0,1\} \\ (a+2)(r-1) & r \in \{2, \ldots, a+1\} \end{cases} =$$

$$(a+1)((a+2)q + r) + \begin{cases} r & r \in \{0,1\} \\ r-(a+2) & r \geq 2 \end{cases} \geq$$

$$(a+1)A'_k - a.$$

It follows that $B'_k = A'_k + ak \geq (a+1)A'_k - a$. Hence, $A'_k \leq k+1$. Now $A'_k \geq k$ for all $k \geq 0$. by the proof of Lemma 2, there are two possibilities:

I. $A'_k = k$, which means $1 \leq k \leq a$,

II. $A'_k = k+1$, which means $a+1 \leq k \leq 2a$.

I. $A'_k = k$. Now, $B'_k = A'_k + ak = (a+1)A'_k$. Since $A_n - A_i = A'_k \leq a$, **(2)** of Theorem 3 implies that there is at most one B_j between A_i and A_n. First assume that there is no such B_j. Then $n - i = A'_k$, so $A_j - A_{j-1} = 1$ for all $i < j \leq n$. Thus by Corollary 3, $B_j - B_{j-1} = a+2$ for all $i < j \leq n$. Hence $B_n - B_i = (a+2)(n-i) = (a+2)A'_k > (a+1)A'_k = B'_k$, which disposes of this case.

Secondly, assume there is a single B_j. Then $A'_k = n - i + 1$. Again by Corollary 3, $B_n - B_i \leq (a+2)(n-i-1) + (a+3) = (a+2)(n-i+1) - (a+1) = (a+1)A'_k + A'_k - a - 1 \leq B'_k - 1 < B'_k$. Hence this case is also rejected.

II. $A'_k = k+1$. Now, $B'_k = A'_k + ak = (a+1)A'_k - a$. As was stated before, $a+1 \leq k \leq 2a$, hence $a+2 \leq A'_k \leq 2a+1$. Thus there is at least one B_j between A_i and A_n, and at most two. A *necessary* condition for there being also B_{j+1} between A_i and A_n, is that $A'_k \geq a+4$.

If there is a single B_j, then $n-i = A'_k - 1$. Therefore $B_n - B_i \geq (a+2)(n-i) = (a+2)(A'_k - 1) = (a+1)A'_k - a + A'_k - 2 = B'_k + (A'_k - 2) \geq B'_k + a > B'_k$. Thus, this option is excluded.

If there are B_j and B_{j+1} then $n-i=A'_k-2$. It follows that $B_n-B_i\geq (a+2)(n-i)=(a+2)(A'_k-2)=(a+1)A'_k-a+(A'_k-(a+4))$. As was mentioned before, $A'_k\geq a+4$, so $B_n-B_i=B'_k$ if and only if $A'_k=a+4$ if and only if $k=a+3$. Now, (A_n,B_n) is a P-position of Γ_2 and (A'_{a+3},B'_{a+3}) is one of its legal moves. Therefore, it cannot lead to another P-position (A_i,B_i). ■

References

1. [BCG1982] E.R. Berlekamp, J.H. Conway and R.K. Guy 1982, *Winning Ways* (two volumes), Academic Press, London.

2. [Con1976] J.H. Conway 1976, *On Numbers and Games*, Academic Press, London.

3. [Cox1953] H.S.M. Coxeter 1953, The golden section, Phyllotaxis and Wythoff's game, *Scripta Math.* **19**, 135–143.

4. [Fra1969] A.S. Fraenkel 1969, The bracket function and complementary sets of integers, *Can. J. Math.* **21**, 6–27.

5. [Fra1982] A.S. Fraenkel 1982, How to beat your Wythoff games' opponents on three fronts, *Amer. Math. Monthly* **89**, 353–361.

6. [Fra1996] A.S. Fraenkel 1996, Scenic trails ascending from sea-level Nim to alpine chess, in: *Combinatorial Games*, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, CA (R.J. Nowakowski, ed.), MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp. 13–42.

7. [FrL1991] A.S. Fraenkel and M. Lorberbom 1991, Nimhoff games, *J. Combin. Theory* (Ser. A) **58**, 1–25.

8. [Sin1981] D. Singmaster 1981, Almost all games are first person games, *Eureka* **41**, 33–37.

9. [Sin1982] D. Singmaster 1982, Almost all partizan games are first person and almost all impartial games are maximal, *J. Combin. Inform. System Sci.* **7**, 270–274.

10. [Wyt1907] W.A. Wythoff 1907, A modification of the game of Nim, *Nieuw Arch. Wisk.* **7**, 199–202.

11. [YaY1967] A.M. Yaglom and I.M. Yaglom 1967, *Challenging Mathematical Problems with Elementary Solutions*, translated by J. McCawley, Jr., revised and edited by B. Gordon, Vol. II, Holden-Day, San Francisco.