Games Played by Boole and Galois*

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Abstract

We define an infinite class of 2-pile subtraction games, where the amount that can be subtracted from both piles simultaneously is an extended Boolean function f of the size of the piles, or a function over GF(2). Wythoff's game is a special case. For each game, the 2nd player winning positions are a pair of complementary sequences. Sample games are presented, strategy complexity questions are discussed, and possible further studies are indicated. The motivation stems from the major contributions of Professor Peter Hammer to the theory and applications of Boolean functions.

Keywords: 2-pile subtraction games, extended Boolean functions, Galois field, integer sequences

1 Introduction

We invented 2-pile Boolean subtraction games to pay tribute to Peter Hammer, in honor of his outstanding scientific achievements, in particular his major contributions to the theory and applications of Boolean and pseudo-Boolean functions. The applications Peter has been contributed to span a very wide spectrum of human activity, including optimization, maximization, minimization, operations research; and lately, medical applications, about which Peter lectured in his captivating invited address at the workshop.

Within the class of 2-player perfect information games without chance moves, we consider games on two piles of tokens (x, y) of sizes x, y, with $0 \le x \le y < \infty$. Their interest stems, inter alia, from the special and important case of Wythoff's game [20]. See also [1], [2], [3], [4], [5], [6], [7], [11], [12], [16], [17], [18], [21].

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For any acyclic combinatorial game, such as 2-pile subtraction games, a position $u = (x, y)$ is labeled N (*Next* player win) if the player moving from u can win; otherwise it's a P-position (*Previous* player win). Denote by $\mathcal{P}$ the set of all P-positions, by $\mathcal{N}$ the set of all N-positions, and by $F(u)$ the set of all (direct) *followers* or *options* of u. It is easy to see that for any acyclic game,

$$u \in \mathcal{P} \quad \text{if and only if} \quad F(u) \subseteq \mathcal{N}\ , \tag{1}$$

$$u \in \mathcal{N} \quad \text{if and only if} \quad F(u) \cap \mathcal{P} \neq \emptyset\ . \tag{2}$$

Indeed, player I, beginning from an N-position, will move to a P-position, which exists by (2), and player II has no choice but to go to an N-position, by (1). Since throughout our games are finite and acyclic, player I will eventually win by moving to a leaf, which is clearly a P-position.

The partitioning of the game's positions into the sets $\mathcal{P}$ and $\mathcal{N}$ is unique for every acyclic combinatorial game without ties.

In our games, two players alternate removing tokens from the piles:

(a) Remove any positive number of tokens from a single pile, possibly the entire pile.

(b) Remove a positive number of tokens from each pile, say k, ℓ, so that $|k-\ell|$ isn't too large with respect to the position (x_1, y_1) moved to from (x_0, y_0), namely, $|k - \ell| < f(x_1, y_1, x_0)$, equivalently:

$$|(y_0 - y_1) - (x_0 - x_1)| = |(y_0 - x_0) - (y_1 - x_1)| < f(x_1, y_1, x_0), \tag{3}$$

where the *constraint function* $f(x_1, y_1, x_0)$ is integer-valued and satisfies:

- Positivity:
$$f(x_1, y_1, x_0) > 0 \;\; \forall y_1 \geq x_1 \geq 0 \;\; \forall x_0 > x_1.$$
- Monotonicity:
$$x_0' < x_0 \Longrightarrow f(x_1, y_1, x_0') \leq f(x_1, y_1, x_0).$$
- Semi-additivity (or generalized triangle inequality) on the P-positions (A_i, B_i) ($A_i \leq B_i$ for all $i \geq 0$), namely: for $n > m \geq 0$,
$$\sum_{i=0}^{m} f(A_{n-1-i}, B_{n-1-i}, A_{n-i}) \geq f(A_{n-m-1}, B_{n-m-1}, A_n).$$

The player making the move after which both piles are empty (a *leaf* of the game) wins; the opponent loses.

Let $S \subset \mathbb{Z}_{\geq 0}$, $S \neq \mathbb{Z}_{\geq 0}$, and $\overline{S} = \mathbb{Z}_{\geq 0} \setminus S$. The *minimum excluded value* of S is

$$\operatorname{mex} S = \min \overline{S} = \text{ least nonnegative integer not in } S.$$

Note that mex of the empty set is 0.

We defined the above class of games in [10], where we proved:

Theorem 1 *Let $\mathcal{S} = \cup_{i=0}^{\infty}(A_i, B_i)$, where, for all $n \in \mathbb{Z}_{\geq 0}$,*

$$A_n = \operatorname{mex}\{A_i, B_i : 0 \leq i < n\}, \tag{4}$$

$B_0 = 0$, and for all $n \in \mathbb{Z}_{>0}$,

$$B_n = f(A_{n-1}, B_{n-1}, A_n) + B_{n-1} + A_n - A_{n-1}. \tag{5}$$

If f is positive, monotone and semi-additive, then $\mathcal{S}$ is the set of P-positions of a general 2-pile subtraction game with constraint function f, and the sequences $A = \cup_{i=1}^{\infty}\{a_i\}$, $B = \cup_{i=1}^{\infty}\{b_i\}$ share the following common features: (i) they partition $\mathbb{Z}_{\geq 1}$; (ii) $b_{n+1} - b_n \geq 2$ for all $n \in \mathbb{Z}_{\geq 0}$; (iii) $a_{n+1} - a_n \in \{1, 2\}$ for all $n \in \mathbb{Z}_{\geq 0}$.

We also showed there that if any of the three conditions of Theorem 3 is dropped, then there are games for which its conclusion fails:

Proposition 1 *There exist 2-pile subtraction games with constraint functions f which lack precisely one of positivity, monotonicity or semi-additivity, such that $\mathcal{S} \neq \mathcal{P}$, where $\mathcal{S} = \cup_{i=0}^{\infty}(A_i, B_i)$, and A_i satisfies (4) ($i \in \mathbb{Z}_{\geq 0}$); $B_0 = 0$, B_n satisfies (5) ($n \in \mathbb{Z}_{>0}$).*

Throughout this paper we consider the case where f is a function over GF(2), or an *extended Boolean* function. The latter is defined (in a seemingly new way) as follows. Variables assume values in $\mathbb{Z}_{\geq 0}$ rather than, as in Boolean Algebra, only in $\{0, 1\}$, and also coefficients and constants are in $\mathbb{Z}_{\geq 0}$. The binary Boolean operators "plus" denoted by $\boxplus$ and "times" denoted by $\boxtimes$ operate bitwise on the vectors that are the binary expansions of the numbers, where $0 \boxplus 0 = 0 \boxtimes 0 = 0 \boxtimes 1 = 1 \boxtimes 0 = 0$, $0 \boxplus 1 = 1 \boxplus 0 = 1 \boxplus 1 = 1 \boxtimes 1 = 1$. These operations are identical to the classical Boolean operations. The special Boolean idiosyncrasy is embodied in the idempotent relation $1 \boxplus 1 = 1$. For example, $1 \boxplus 3 = 2 \boxplus 3 = 3 \boxplus 3 = 3$, $5 \boxplus 6 = 7$.

We have $a \boxplus a = a \boxtimes a = a$ for all $a \in \mathbb{Z}_{\geq 0}$, and

$$1 \boxtimes a = \begin{cases} 0 & \text{if } a \text{ is even} \\ 1 & \text{if } a \text{ is odd} \end{cases}.$$

Further, $\boxplus$ and $\boxtimes$ are both associative and commutative, since the bitwise operations are. The same argument shows that also the two distributive laws hold: $a \boxtimes (b \boxplus c) = (a \boxtimes b) \boxplus (a \boxtimes c)$ and $a \boxplus (b \boxtimes c) = (a \boxplus b) \boxtimes (a \boxplus c)$.

For defining extended Boolean complements, let $C(a)$ denote the 1's-complement of a, beginning from the most-significant 1-bit of a. For $b \geq a$, let $C_b(a) = C(2^{\lfloor \log b \rfloor + 1} + a)$. Then $C_a(a) = C(2^{\lfloor \log a \rfloor + 1} + a) = C(1a) = C(a)$. The Boolean rule of involution $C^2(x) = C(C(x)) = x$ has then the following form in our extended Boolean algebra.

Proposition 2 *For all $a \in \mathbb{Z}_{\geq 0}$, $C_a^2(a) = C_a(C(a)) = a$.*

Proof. $C_a^2(a) = C(2^{\lfloor \log a \rfloor + 1} + C(a))$, which is the 1's complement of $C(a)$ beginning from a 1-bit left-adjacent to the most significant bit of $C(a)$. Thus $C_a^2(a) = C(1C(a)) = a$. ■

Table 1. The first few P-positions for $\mathbf{G}_1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	0	1	3	4	6	7	8	10	11	12	13	14	15	17	18	19	20
B_n	0	2	5	9	16	24	32	43	55	67	81	95	111	128	146	166	186

2 Bach, Boole, Escher, Galois, Gödel

Consider the 2-pile subtraction game denoted $\mathbf{G}_1$, subject to conditions (a), (b) above, with constraint function $f(x_1, y_1, x_0) = x_1 \boxplus 1$. Positivity and monotonicity hold trivially, and semi-additivity follows from $(A_{n-1} \boxplus 1) + (A_{n-2} \boxplus 1) \geq A_{n-2} \boxplus 1$. Thus by (5),

$$B_n = (A_{n-1} \boxplus 1) + B_{n-1} + A_n - A_{n-1}. \qquad (6)$$

The first few P-positions (A_n, B_n) of $\mathbf{G}_1$, where A_n satisfies (4) and B_n satisfies (6), are depicted in Table 1.

The P-positions are the key for winning. From any position not in Table 1, such as $(4, 8)$, there is a legal move leading back into the table. In fact, $(4, 8) \to (3, 5)$ is legal, since $(8 - 4) - (5 - 3) = 2 < 3 \boxplus 1 = 3$. The position $(4, 8)$ is an N-position, and in order to win, the Next player will make the move to $(3, 5)$. But $(4, 9)$, which is in the table, cannot be moved legally into any other table position. It's a Previous player position, i.e., a P-position.

On p. 73 of [15], the reader is asked to characterize the following sequence:

$$B'_{n\geq 0} = \{1,\ 3,\ 7,\ 12,\ 18,\ 26,\ 35,\ 45,\ 56, \ldots\}.$$

Answer: the sequence $\{2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11, \ldots\}$ constitutes the set of differences of consecutive terms of B'_n, as well as the complement with respect to $\mathbb{Z}_{>0}$ of B'_n. For our purposes it is convenient to preface 0 to the second sequence, so we define

$$A'_{n\geq 0} = \{0,\ 2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11, \ldots\},$$

Comparing the sequences A'_n, A_n and B'_n, B_n, there doesn't seem to be a clear connection. But from (6),

$$B_n - B_{n-1} = A_n + (A_{n-1} \boxplus 1) - A_{n-1} = \begin{cases} A_n + 1 & \text{if } A_{n-1} \text{ even} \\ A_n & \text{if } A_{n-1} \text{ odd} \end{cases}, \qquad (7)$$

so the Hofstadter property that A_{n-1} is the difference between B_n and B_{n-1}, in addition to being its complement, is almost retained for the P-positions of the game $\mathbf{G}_1$. Our first Boolean game is thus related to Bach, Escher and Gödel.

Table 2. The first few P-positions for $\mathbf{G}_2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	0	1	3	5	6	7	9	10	11	12	14	15	16	17	18	19	20
B_n	0	2	4	8	13	21	29	38	50	61	76	92	107	125	142	162	181

Before continuing with another Boolean game, let us make a short detour via a game for which the constraint function is computed over the field GF(2), for which $0 \oplus 0 = 1 \oplus 1 = 0 \otimes 0 = 0 \otimes 1 = 1 \otimes 0 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1 \otimes 1 = 1$. For example, $3 \oplus 6 = 5$, $3 \otimes 6 = 2$. Note that $a \otimes b = a \boxtimes b$ for all nonnegative integers a, b; we use $a \boxtimes b$ in the Boolean context, and $a \otimes b$ in Galois-type formulas.

The constraint function for $\mathbf{G}_2$ is $f(x_1, y_1, x_0) = x_1 \oplus 1$. Positivity and monotonicity are trivially satisfied, and semi-additivity follows from $(A_{n-1} \oplus 1) + (A_{n-2} \oplus 1) \geq A_{n-2} \oplus 1$. From (5), $B_n = (A_{n-1} \oplus 1) + B_{n-1} + A_n - A_{n-1}$. Thus,

$$B_n - B_{n-1} = A_n + (A_{n-1} \oplus 1) - A_{n-1} = \begin{cases} A_n + 1 & \text{if } A_{n-1} \text{ even} \\ A_n - 1 & \text{if } A_{n-1} \text{ odd} \end{cases}, \tag{8}$$

analogously to the behavior of $\mathbf{G}_1$. A prefix of the P-positions for this *Galois* game is shown in Table 2.

Prior to generalizing the games $\mathbf{G}_1$ and $\mathbf{G}_2$, it is helpful to prove the following auxiliary result.

Proposition 3 *Let $b \geq a \geq 0$. Then $a \boxplus b = b + (a \boxtimes C(b)) = a + (b \boxtimes C_b(a))$, with extrema*

$$a \boxplus b = \begin{cases} b & \textit{if } \; a \boxtimes b = a \\ b + a & \textit{if } \; a \boxtimes b = 0 \end{cases}. \tag{9}$$

Similarly, $a \oplus b = b + (a \otimes C(b)) - (a \otimes b) = a + (b \otimes C_b(a)) - (a \otimes b)$, with extrema

$$a \oplus b = \begin{cases} b - a & \textit{if } \; a \otimes b = a \\ b + a & \textit{if } \; a \otimes b = 0 \end{cases}.$$

Proof. For $b \geq a$, $a \boxplus b$ has 1-bits precisely where b has, augmented by 1-bits at positions where a has 1-bits and b doesn't, i.e., at positions where $a \boxtimes C(b)$ has 1-bits. Similarly, it has 1-bits precisely where a has, augmented by 1-bits

Table 3. The first few P-positions for $\mathbf{G}_3$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	0	1	2	3	5	6	7	9	10	11	13	14	15	16	18	19	20
B_n	0	4	8	12	17	25	33	42	54	66	79	95	111	127	148	168	188

at positions where b has 1-bits and a doesn't, i.e., at $a + (b \boxtimes C_b(a))$, since $b \geq a$. The same argument holds for $a \oplus b$, but we have to subtract the 1-bits co-occurring in a and b. ■

Theorem 2 *Let $k \in \mathbb{Z}_{>0}$. For the two-pile game with constraint function $f(x_1, y_1, x_0) = x_1 \boxplus k$, we have $\mathcal{P} = \cup_{n=0}^{\infty}(A_n, B_n)$, where A_n, B_n are given by* (4), (5) *respectively, and for $n \geq 1$,*

$$B_n - B_{n-1} = \begin{cases} A_n + (C(A_{n-1}) \boxtimes k) & \text{if } A_{n-1} \geq k \\ A_n - A_{n-1} + k + (A_{n-1} \boxtimes C(k)) & \text{if } A_{n-1} \leq k \end{cases}. \tag{10}$$

A similar result holds for the constraint function $f(x_1, y_1, x_0) = x_1 \oplus k$, namely, for $n \geq 1$,

$$B_n - B_{n-1} = \begin{cases} A_n + (C(A_{n-1}) \otimes k) - (A_{n-1} \otimes k) & \text{if } A_{n-1} \geq k \\ A_n - A_{n-1} + k + (A_{n-1} \otimes C(k)) & \\ -(A_{n-1} \otimes k) & \text{if } A_{n-1} \leq k \end{cases}. \tag{11}$$

Proof. As before, positivity, monotonicity and semi-additivity are easily seen to hold for both $x_1 \boxplus k$ and $x_1 \oplus k$. By Theorem 1 we get, for $x_1 \boxplus k$,

$$B_n - B_{n-1} = A_n + (A_{n-1} \boxplus k) - A_{n-1},$$

and (10) follows directly from the first part of Proposition 3.

An analogous argument proves the validity of (11), using the second part of Proposition 3. ■

Note that for the special case $k = 1$, (10) implies (7), and (11) implies (8). The first few P-positions of $\mathbf{G}_3$ and $\mathbf{G}_4$ for $k = 3$ are displayed in Tables 3 and 4 respectively, where $f(x_1, y_1, x_0) = x_1 \boxplus 3$ for $\mathbf{G}_3$, and $f(x_1, y_1, x_0) = x_1 \oplus 3$ for $\mathbf{G}_4$.

3 Games Boole and Galois Played Together

What happens when Boole and Galois join together in a game? Here's a special case of a 2-pile subtraction game they liked to play, with the constraint function

Table 4. The first few P-positions for $\mathbf{G}_4$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	0	1	2	3	5	6	8	10	12	13	14	15	16	17	19	20	21
B_n	0	4	7	9	11	18	25	38	49	65	80	94	107	127	147	164	188

Table 5. The first few P-positions for $\mathbf{G}_5$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	0	1	3	4	5	6	8	9	10	11	12	14	15	16	17	18	19
B_n	0	2	7	13	23	43	90	174	342	692	1396	2799	5585	11185	22355	44695	89373

$f(x_1, y_1, x_0) = (x_1 \oplus y_1) \boxplus 1$, containing both a Boolean and Galois-type operator. Positivity and monotonicity are again immediate. Semi-additivity is implied by $((A_{n-1} \oplus B_{n-1}) \boxplus 1) + ((A_{n-2} \oplus B_{n-2}) \boxplus 1) \geq ((A_{n-2} \oplus B_{n-2}) \boxplus 1)$. The first few P-positions for this game, $\mathbf{G}_5$, are depicted in Table 5, where, by (5), $B_n = B_{n-1} + A_n - A_{n-1} + ((A_{n-1} \oplus B_{n-1}) \boxplus 1)$.

Homework. (i) Compute the P-positions for the "complementary" Boole-Galois collaborative game with the constraint function $f(x_1, y_1, x_0) = (x_1 \boxplus y_1) \oplus 1$.
(ii) Let $k \in \mathbb{Z}_{>0}$. Compute the P-positions for the game with constraint function $f(x_1, y_1, x_0) = (x_1 \oplus y_1) \boxplus k$; and also for the game with $f(x_1, y_1, x_0) = (x_1 \boxplus y_1) \oplus k$.

We have thus learned how to win even when Boole and Galois contribute jointly to the constraint function. Hmm...Could Boole and Galois have played together? Well, George Boole (1815-64) lived in England, and Évariste Galois (1811–32) in France. Of course Galois became a genius because of the mathematical games he played with Boole for his last 13 years, from age 8, when he decided to teach the 4-year old Boole, across the English Channel, via the 19-th century Channel-Internet!

Table 6. The first few values of $\mathcal{S}$ for $\mathbf{G}_6$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	0	1	3	4	5	6	8	9	10	11	12	13	14	16	17	18	19
B_n	0	2	7	15	21	29	45	55	67	79	95	109	125	157	175	195	215

4 Hammer Wins Even When the Theory Fails!

Not only did Peter Hammer give us wonderful new insights into the theory and applications of Boolean functions; he also taught us the important art of selecting profitable research topics. Peter didn't choose to work in esoteric areas of mathematics, but, instead, on Boolean functions, with the rich abundance of applications he managed to squeeze from the theory he has developed. I'll demonstrate, however, a deeper reason for Peter's clever embrace of Booleanity!

Consider the Boolean game $\mathbf{G}_6$ with the constraint function $f(x_1, y_1, x_0) = x_1 \boxplus x_0$. It is clearly positive. Semi-additivity is implied by (9): $(A_{n-1} \boxplus A_n) + (A_{n-2} \boxplus A_{n-1}) \geq A_n + A_{n-1} > A_n + A_{n-2} \geq A_n \boxplus A_{n-2}$. But the counterexample $2 \boxplus 5 = 7 > 2 \boxplus 6 = 6$ shows that f is not monotone. We can still compute the first few positions of $\mathcal{S} = \cup_{n=0}^{\infty}(A_n, B_n)$, displayed in Table 6, where A_n, B_n are given by (4), (5) respectively, so $B_n = (A_{n-1} \boxplus A_n) + B_{n-1} + A_n - A_{n-1}$. But Theorem 1 fails to tell us whether they are or are not P-positions.

In the proof of Theorem 1 in [10] it is first shown, independently of monotonicity, that

$$A_n > A_{n-1} \tag{12}$$

for all $n \in \mathbb{Z}_{>0}$, and

$$B_n - A_n > B_m - A_m \geq 0 \text{ for all } n > m \geq 0. \tag{13}$$

Monotonicity is used in precisely two places. The first one is in showing that A and B are *complementary* sets of integers, i.e., $A \cup B = \mathbb{Z}_{\geq 1}$, and $A \cap B = \emptyset$, where $A = \cup_{n=1}^{\infty} A_n$, $B = \cup_{n=1}^{\infty} B_n$. This is done as follows: if $A_n = B_m$, then $n > m$ implies that A_n is the mex of a set containing $B_m = A_n$, a contradiction to the mex definition; and $1 \leq n \leq m$ is impossible since

$$\begin{aligned} B_m &= f(A_{m-1}, B_{m-1}, A_m) + B_{m-1} - A_{m-1} + A_m \\ &\geq f(A_{m-1}, B_{m-1}, A_n) + B_{n-1} - A_{n-1} + A_n \\ &\quad \text{(by (12), (13) and monotonicity)} \\ &> A_n \text{ (by positivity).} \end{aligned}$$

Though monotonicity fails for the present game, the proof of the complementarity of A and B can be completed in a simple manner for $\mathbf{G}_6$. For $m \geq n \geq 1$,

$$
\begin{aligned}
B_m &= (A_{m-1} \boxplus A_m) + B_{m-1} - A_{m-1} + A_m \\
&\geq 2A_m + B_{m-1} - A_{m-1} \text{ (by (9))} \\
&\geq 2A_n + B_{n-1} - A_{n-1} \text{ (by (12), (13))} \\
&> A_n.
\end{aligned}
$$

The second place where monotonicity is used in the proof of Theorem 1 is towards the end, where we conclude

$$f(A_m, B_m, A_{m+1}) \leq f(A_m, B_m, A_n) \tag{14}$$

for all $0 \leq m < n$. As we pointed out, $2 \boxplus 5 = 7 > 2 \boxplus 6 = 6$, so without monotonicity, we don't seem to be able to get (14). We use an auxiliary result.

Proposition 4 *Let a, b, c be integers satisfying $0 \leq a < b \leq c,\ \ b - a \in \{1, 2\}$. Then $a \boxplus b \leq a \boxplus c$.*

Proof. (A) Suppose $b - a = 1$. Then a, b have different parities. If a is even, then, by (9), $a \boxplus b = a \boxplus (a+1) = a + 1 = b \leq c \leq a \boxplus c$. If a is odd, we may write,

$$a = \sum_{i=0}^{k-1} 2^i + \sum_{i \geq k+1} \varepsilon_i 2^i, \tag{15}$$

where $\varepsilon_i \in \{0, 1\}$ ($i \geq k+1$), and $k \geq 1$. Then $b = a + 1 = 2^k + \sum_{i \geq k+1} \varepsilon_i 2^i$. Further,

$$a \boxplus b = \sum_{i=0}^{k} 2^i + \sum_{i \geq k+1} \varepsilon_i 2^i. \tag{16}$$

Since $c \geq b$, we can write $c = 2^k + \sum_{i \geq k+1} \varepsilon_i 2^i + s$ for some $s \geq 0$. Hence $a \boxplus c = \sum_{i=0}^{k} 2^i + \sum_{i \geq k+1} \varepsilon_i 2^i + s \geq a \boxplus b$.

(B) Suppose $b - a = 2$. Then a, b have the same parity. We first consider the case where both are odd. Then we may assume that a is given by (15), so $b = a + 2 = 1 + 2^k + \sum_{i \geq k+1} \varepsilon_i 2^i$. Thus $a \boxplus b$ is again given by (16), so we conclude, as in (A) above, $a \boxplus b \leq a \boxplus c$.

Now consider the case where a, b are both even. Since $b - a = 2$, one of a, b has the form $4\ell + 2$, whereas the other is divisible by 4. If $a = 4\ell$, $b = 4\ell + 2$, then $a \boxplus b = b \leq c \leq a \boxplus c$. In the other case, a is twice the right hand side of (15), and the proof proceeds in a straightforward way as in case (A), but a, b are multiplied by 2. We omit the details. ■

In [10] it was shown, based on the complementarity of A, B, that $A_n - A_{n-1} \in \{1, 2\}$ for all $n \geq 1$. By Proposition 4 we thus see that (14) holds also for $\mathbf{G}_6$.

In conclusion, $\mathcal{S} = \mathcal{P}$ for $\mathbf{G}_6$, where the first few entries of $\mathcal{S}$ are displayed in Table 6.

Thus, although $\mathbf{G}_6$ fails to satisfy the hypotheses of the general Theorem 1, it nevertheless enjoys its conclusions. Peter Hammer demonstrated the merit of picking a research area where conclusions are valid even when the theory fails!

5 Epilogue

We have presented an assortment of Boolean and Galois 2-player subtraction games on 2 piles of tokens. There are two types of moves: either remove any positive number of tokens from a single pile, or else, take $k > 0$ from one and $\ell > 0$ from the other, subject to $|k - \ell| < f$, where f is a suitable extended Boolean or GF(2)-type function. These games were motivated by Peter Hammer's fascinating work with Boolean functions.

The generalized Wythoff game [6] is a special case of the family of games considered here, namely, the case $f = c$, where c is a positive integer constant. (The case $c = 1$ is the original game as defined by Wythoff.) It has the property that a polynomial strategy can be given by using a special numeration system, and noting that the A_n members are characterized by ending in an even number of 0s in that representation, and the B_n being their left shifts.

Some of the remaining open questions:
(i) Determine subsets of 2-pile subtraction games for which the indicated strategy is polynomial. (The computation of the P-positions presented here is exponential in their *succinct* (logarithmic) input size.)
(ii) Extend the games in a natural way to multi-pile games. This seems to be difficult for Wythoff's game, for which I have a conjecture; see [8] §6(2), [14] Problem 53, [9] §5, [19].
(iii) Compute the Sprague-Grundy function for the games or a subset of them. A polynomial algorithm for this would permit to play *sums* of such games efficiently. Seems difficult for Wythoff's game.
(iv) Compute a strategy for the games when played in misère version, i.e., the player making the last move loses. This is easy for Wythoff's game. See [1], ch. 13.
(v) Computation of complexities of P-positions sequences; for example, Kolmogorov-, program-, subword-, palindrome-, squares-complexities. For a related class of 2-pile subtraction games the subword complexity was computed in [13].

Peter Hammer died tragically on Dec. 27, 2006, four days after his 70th birthday. This paper was written in happier days, when Peter was bristling with scientific activity. We preferred to leave the setting happy, rather than turning it into a eulogy, because Peter was a person radiating optimism and happiness, and I cannot but remember him that way.

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