Unbounded Iterations of Floor Functions and the Flora Game

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Abstract

Partition the set of games into subsets *CompGames* and *PrimGames*, precipitating a new 4-pile take-away game *Flora*. Let $\varphi = (1 + \sqrt{5})/2$ denote the golden section. We investigate relationships between unbounded iterations of the floor function applied to various combinations of φ and φ^2. We use them to formulate an algebraic polynomial-time winning strategy for Flora, and also present recursive, arithmetic and morphic winning strategies for it. The arithmetic one is based on the Fibonacci numeration system. The four strategies differ in their computational efficiencies. We further show how to generate the sequences induced by the iterations using morphisms and and characterize them using the Fibonacci numeration system. We also exhibit an infinite array of such sequences.

Keywords: floor function, integer part function, combinatorial game theory, Fibonacci numeration system

1 Introduction

As customary, we denote by $\lfloor x \rfloor$ the integer part of x, commonly known as the *floor function*. It is the largest integer not exceeding x. Let $\varphi = (1 + \sqrt{5})/2$ denote the golden section.

Two topics motivate this work. On the one hand, we wish to study what happens when we keep iterating the floor function with either φ or φ^2 in various ways. Are any interesting relationships between them discernible even after an unbounded number of iterations, or total chaos takes over?

On the other hand, we aim at studying the class of impartial take-away games. This class appears to be partitioned into two disjoint subclasses: those that are easy to generalize to more than one or two piles, and those for which

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this seems to be very hard. A well-known representative of the former is Nim [2], and of the latter Wythoff's game [5]. Some progress in generalizing Wythoff to multiple piles was recently made. See [9], [20], [19]. Three-pile games that are *extensions* rather than generalizations of Wythoff were also given recently [8], [4].

Here we consider an extension of Wythoff to four piles. The efforts in defining a "right" extension and particularly in proving the validity of the winning strategy are considerably greater than those for three-pile extensions. We present four formulations of the winning strategy.

In §2 we investigate the unbounded iterations and come up with a wealth of relationships and identities. In §3 we define the subclasses *CompGames* and *PrimGames*, which motivate the definition of the 4-pile game, dubbed Flora. In §3.1 we formulate an algebraic polynomial-time winning strategy for the game, based on the results derived in §2. In §3.2 we formulate a recursive winning strategy which is deceptively simple, but appears to have higher computational complexity. We end in §3.3 with a polynomial-time arithmetic winning strategy, based on the Fibonacci numeration system. In §4 we indicate how to generate sequences induced by iterations of the floor function using morphisms. We apply it to one of the sequences in §4.1. In §4.2 we present our fourth, morphic, formulation of the winning strategy of the Flora game, which is also polynomial-time. In §4.3 we use results from §3.3 and §1 and make minor use of the language of §4, to characterize the representations of general cases of the sequences playing a major role in the algebraic formulation of the winning strategy of Flora. In §5 we show, by means of an example, how to produce infinite complementary arrays using sequences induced by the iterations. In the final Epilogue we wrap up and indicate natural further directions of research.

Let $g(n,1) = \lfloor n\varphi \rfloor$, $h(n,1) = \lfloor n\varphi^2 \rfloor$, $u(n,1) = g(n,1)$, $v(n,1) = h(n,1)$, and for $k \geq 2$,

$$
\begin{aligned}
g(n,k) &= \lfloor g(n,k-1)\varphi \rfloor, & h(n,k) &= \lfloor h(n,k-1)\varphi \rfloor, \\
u(n,k) &= \lfloor g(n,k-1)\varphi^2 \rfloor, & v(n,k) &= \lfloor h(n,k-1)\varphi^2 \rfloor.
\end{aligned}
$$

Let $\Delta_g(n,k) = g(n+1,k) - g(n,k)$, $\Delta_h(n,k) = h(n+1,k) - h(n,k)$. For technical reasons we put

$$ g(n,0) = n, \qquad h(n,0) = g(n,1). $$

Further, let $F_{-1} = 1$, $F_0 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 1)$ be the Fibonacci sequence.

Notation 1. For $k \geq 0$, let $G_k = \cup_{n=1}^{\infty} g(n,k)$, $H_k = \cup_{n=1}^{\infty} h(n,k)$, $U_k = \cup_{n=1}^{\infty} u(n,k)$, $V_2 = \cup_{n=1}^{\infty} v(n,2)$, $G_k - s = \cup_{n=1}^{\infty} (g(n,k) - s)$ (subtracting s from every element of G_k).

In particular, $G_0 = \mathbb{Z}_{\geq 1}$, and $H_0 = G_1 = U_1$.

2 Identities

After multiplying by the irrational φ and then throwing out the fractional part for an unbounded number of times, one might expect complete chaos among relationships involving $g(n,k)$, $h(n,k)$, $u(n,k)$ and $v(n,k)$. It is thus surprising that there are many striking identities and relationships among them. Our purpose in this section is to prove a selection of them. Throughout it is useful to think of n as running over all values in $[1,\infty)$ for every fixed, though arbitrary, value k.

Theorem 1. *For every $k \in \mathbb{Z}_{\geq 1}$ and all $n \in \mathbb{Z}_{\geq 1}$ the following holds:*

  (a)   *The $k+1$ sequences $G_k, H_k, H_{k-1}, \ldots, H_2, H_1$ partition $\mathbb{Z}_{\geq 1}$.*

  (b) $u(n, k+1) = g(n,k) + g(n, k+1) = g(n, k+2) + 1.$

  (c) $h(n,k) = g(n, k+1) + F_{k-1}.$

  (d) $u(n, k+1) = h(n, k+1) - F_k + 1 = g(n, k+2) + 1.$

  (e) (e1) $h(n, k+1) - h(n,k) = g(n,k) + F_{k-2} - 1.$

      (e2) $h(n, k+1) - g(n, k+1) = g(n,k) + F_k - 1.$

      (e3) $\lfloor u(n, k+1)\varphi \rfloor = u(n, k+2) + 1.$

  (f) (f1)   *Let*

$$S_1 = \{n \in \mathbb{Z}_{\geq 1} : \Delta_g(n,1) = F_0\}, \quad S_2 = \{n \in \mathbb{Z}_{\geq 1} : \Delta_g(n,1) = F_1\}.$$

      Then S_1 and S_2 split $\mathbb{Z}_{\geq 1}$; and each of S_1 and S_2 is an infinite set.

      (f2)   *For all $k \in \mathbb{Z}_{\geq 1}$: $\Delta_g(n,k) = F_{k-1}$ for all $n \in S_1$ and $\Delta_g(n,k) = F_k$ for all $n \in S_2$.*

      (f3)   (i) *If $\Delta_g(n+1, k) = F_{k-1}$ for some $n \in \mathbb{Z}_{\geq 1}$, then $\Delta_g(n,k) = \Delta_g(n+2, k) = F_k$. (ii) If $\Delta_g(n+1, k) = \Delta_g(n+2, k) = F_k$ for some $n \in \mathbb{Z}_{\geq 1}$, then $\Delta_g(n,k) = \Delta_g(n+3, k) = F_{k-1}$.*

      (f4) $\Delta_g(n,k) = \Delta_h(n, k-1) \in \{F_{k-1}, F_k\}$, *and each of F_{k-1} and F_k is assumed for infinitely many n.*

      (f5)   (i) $\Delta_g(0,k) = 1,$ (ii) $\Delta_h(0,k) = F_{k-1} + 1.$

  (g) $g(h(n,1), k) = h(n, k+1)$ (due to Lior Goldberg).

  (h) (h1) $(G_2 + 2) \subset G_1.$

      (h2) $G_2 \cup (G_2 + 2) = G_1.$

      (h3) $U_2 \subset (G_1 - 2) \subset G_1 \cup U_2.$

      (h4) $(V_2 - 1) \subset G_2.$

We begin by recalling some elementary properties of the floor function. Let x, y be any real numbers. Denote by $\{x\}$ the fractional part of x, so $x = \lfloor x \rfloor + \{x\}$. Then:

- $0 \leq \{x\} < 1$, $x - 1 < \lfloor x \rfloor \leq x$. Replacing x by $-x$, $-x - 1 < \lfloor -x \rfloor \leq -x$, hence $-1 \leq \lfloor x \rfloor + \lfloor -x \rfloor \leq 0$ and $\lfloor x \rfloor + \lfloor -x \rfloor = 0$ if and only if x is an integer. For example, $\lfloor \varphi \rfloor = 1$, $\lfloor -\varphi \rfloor = -2$, $\lfloor \varphi \rfloor + \lfloor -\varphi \rfloor = -1$; and $\varphi^2 = \varphi + 1$ implies $\{\varphi\} = 1 - \varphi$.

- $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$. This follows immediately from $\lfloor x + y \rfloor = \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$.

Notation 2. *For any set or sequence T of reals, s any real, $T + s := \{t + s : t \in T\}$.*

Lemma 1. (i) *Let $s \in \mathbb{Z}$. Each of the sequences $G_k + s, H_k + s, U_k + s, V_2 + s$ is strictly increasing for every $k \geq 1$.*
(ii) *The sequences G_k, H_k split G_{k-1} for every $k \geq 1$.*

Proof. (i) Follows from the fact that $\varphi^2 = \varphi + 1 > \varphi > 1$.
(ii) Since $\varphi^{-1} + \varphi^{-2} = 1$, the sequences G_1 and H_1 split $\mathbb{Z}_{\geq 1} = G_0$ (see e.g., [5], §3), so the result holds for $k = 1$. For any $k \geq 1$, assume that G_k, H_k split G_{k-1}. Then

$$
\begin{aligned}
G_{k+1} \cup H_{k+1} &= \bigcup_n (g(n, k+1) \cup h(n, k+1)) = \bigcup_n (\lfloor g(n, k)\alpha \rfloor \cup \lfloor h(n, k)\alpha \rfloor) \\
&= \bigcup_n \lfloor g(n, k-1)\alpha \rfloor \text{ (by induction)} = \bigcup_n g(n, k) = G_k. \quad \blacksquare
\end{aligned}
$$

Note. In our applications, $s \in \{0, -1, -2, 2\}$, most often 0.

Proof of Theorem 1 (a). We noted that G_1 and H_1 split $\mathbb{Z}_{\geq 1}$. Suppose that $G_k, H_k, H_{k-1}, \ldots, H_2, H_1$ partition $\mathbb{Z}_{\geq 1}$. Then $G_{k+1}, H_{k+1}, H_k, \ldots, H_2, H_1$ partition $\mathbb{Z}_{\geq 1}$, since G_{k+1}, H_{k+1} split G_k by Lemma 1. $\quad \blacksquare$

Note. It follows from Lemma 1 (or from **(a)**) that for any positive integers m, n, $\lfloor m\varphi \rfloor \neq \lfloor n\varphi^2 \rfloor$. This property will be referred to in the sequel as *disjointness*.

Proof of Theorem 1 (b). By definition,

$$
g(n, k+2) = \lfloor g(n, k+1)\varphi \rfloor = \lfloor \lfloor g(n, k)\varphi \rfloor \varphi \rfloor \leq \lfloor g(n, k)\varphi^2 \rfloor.
$$

By disjointness, $\lfloor g(n, k)\varphi^2 \rfloor \geq g(n, k+2) + 1$. Conversely, multiply $g(n, k)\varphi < g(n, k+1) + 1$ by φ to get, $g(n, k)\varphi^2 < (g(n, k+1) + 1)\varphi$, hence $\lfloor g(n, k)\varphi^2 \rfloor \leq \lfloor (g(n, k+1) + 1)\varphi \rfloor$. By disjointness this inequality is strict, so

$$
\lfloor g(n, k)\varphi^2 \rfloor \leq \lfloor (g(n, k+1) + 1)\varphi \rfloor - 1 = g(n, k+2) + \lfloor \varphi \rfloor = g(n, k+2) + 1.
$$

On the other hand, $\lfloor g(n, k)\varphi^2 \rfloor = \lfloor g(n, k)(\varphi + 1) \rfloor = g(n, k) + g(n, k+1)$. $\blacksquare$

Lemma 2. *For every $k \in \mathbb{Z}_{\geq 1}$,*
   (i) $\lfloor F_{k-1}\varphi \rfloor \in \{F_k - 1, F_k\}$.
   (ii) $\lfloor F_{2k-2}\varphi^2 \rfloor = F_{2k} - 1$, $\lfloor F_{2k-1}\varphi^2 \rfloor = F_{2k+1}$.

Proof. (i) The ratios F_k/F_{k-1} are the convergents of the simple continued fraction expansion of $\varphi = [1, 1, 1, \ldots]$. Therefore $|F_{k-1}\varphi - F_k| < F_{k-1}^{-1}$ (see e.g., [12], ch. 10), so $F_{k-1}\varphi - F_k = \delta$, where $-F_{k-1}^{-1} < \delta < F_{k-1}^{-1}$. Thus $\lfloor F_{k-1}\varphi \rfloor = F_k + \lfloor \delta \rfloor$. The result follows if $|\delta| < 1$, which is the case for all $k \geq 1$, since $F_{k-1} \geq F_0 = 1$.

(ii) The ratios F_{k+2}/F_k are the convergents of the simple continued fraction expansion of $\varphi^2 = [2, 1, 1, 1, \ldots]$. In fact, $F_{2k+1}/F_{2k-1} < \varphi^2 < F_{2k}/F_{2k-2}$. This follows easily from [12], ch. 10. Then $F_{2k-1}\varphi^2 - F_{2k+1} = \delta$, where $0 < \delta < F_{2k-1}^{-1}$, hence $\lfloor F_{2k-1}\varphi^2 \rfloor = F_{2k+1}$, since $0 < \delta < 1$ for all $k \geq 1$. Similarly, $F_{2k-2}\varphi^2 - F_{2k} = \delta$, where $-F_{2k-2}^{-1} < \delta < 0$. Thus $\lfloor F_{2k-2}\varphi^2 \rfloor = F_{2k} - 1$, since $-1 < \delta < 0$ for all $k \geq 1$.   ∎

Lemma 3. *For every $n \in \mathbb{Z}_{\geq 1}$, $h(n, 2) - g(n, 3) = 2$.*

Proof. In Lemma 9 of [8] we proved the special case $k = 2$ of **(d)**, namely $h(n, 2) = u(n, 2) + 1$. Thus $h(n, 2) - g(n, 3) = h(n, 2) - g(n, 3) + 1$. Clearly $g(n, 3) = \lfloor \lfloor \lfloor n\varphi \rfloor \varphi \rfloor \varphi \rfloor \leq \lfloor g(n, 1)\varphi^2 \rfloor = h(n,2)$. But this inequality is strict by disjointness. Thus $h(n, 2) - g(n, 3) \geq 2$.

Conversely, multiply the inequality $g(n, 1)\varphi < g(n, 2) + 1$ by φ, to get $g(n, 1)\varphi^2 < (g(n, 2) + 1)\varphi$. Therefore $\lfloor g(n, 1)\varphi^2 \rfloor \leq \lfloor (g(n, 2) + 1)\varphi \rfloor$. Again by disjointness, this inequality is strict. Now Lemma 9 of [8] asserts $\lfloor g(n, 1)\varphi^2 \rfloor = h(n, 2) - 1$. Therefore, $h(n, 2) < \lfloor (g(n, 2) + 1)\varphi \rfloor + 1$, so $h(n, 2) - g(n, 3) < \lfloor g(n, 2)\varphi + \varphi \rfloor - g(n, 3) + 1 \leq 3$. Hence $h(n, 2) - g(n, 3) \leq 2$.   ∎

Notation 3. For any positive integer N, denote by $R(N)$ the representation of N in the Fibonacci numeration system. It has the form $R(N) = (d_m, \ldots, d_0)$ if $N = \sum_{i=0}^{m} d_i F_i$, where $d_i \in \{0, 1\}$, $d_i = 1 \implies d_{i-1} = 0$, $i \geq 1$ [6]. The *position* of a representation is the subscript of d_i. Thus, d_0 is in position 0, d_1 in position 1, etc.

Proof of Theorem 1 (c). For $k = 1$, this is Lemma 5 of [8], Sect. 5. For $k = 2$, it is Lemma 3 above. We proceed by induction on k for arbitrary $n \in \mathbb{Z}_{\geq 1}$. Suppose that $h(n, k) = g(n, k + 1) + F_{k-1}$ for some fixed $k \geq 2$. Multiply by φ and take the floor of both sides. This gives, by Lemma 2, $h(n, k + 1) = \lfloor (g(n, k+1) + F_{k-1})\varphi \rfloor \leq g(n, k+2) + \lfloor F_{k-1}\varphi \rfloor + 1 \leq g(n, k+2) + F_k + 1$. Now [8] Sect. 6 implies that $R(g(n, 2))$ ends in 01 for every $n \in \mathbb{Z}_{\geq 1}$. By Lemma 1, the same holds for $g(n, k)$ and $h(n, k)$ for every $k \geq 3$ (but it does not hold for $h(n, 2)$). Since $R(F_k)$ ends in 00 for $k \geq 2$, $R(g(n, k+2) + F_k)$ also ends in 01 for $k \geq 2$, and so does $h(n, k+1)$ for $k \geq 2$. But $R(g(n, k+2) + F_k + 1)$ ends in 10. Hence $h(n, k + 1) = g(n, k + 2) + F_k$.   ∎

Proof of Theorem 1 (d). From **(b)** and **(c)**, $u(n, k+1) = g(n, k+2)+1 = h(n, k + 1) - F_k + 1$. The second follows once more from **(c)**.   ∎

We note that inspection shows that **(d)** does not hold for $k < 1$.

Proof of Theorem 1 (e1). Subtracting **(c)** from **(c)** with k replaced by $k+1$ gives $h(n, k+1) - h(n, k) = g(n, k+2) - g(n, k+1) + F_{k-2}$. Substituting the value of $g(n, k+2)$ from **(b)** yields the desired result.　∎
(e2) Follows from **(e1)**, where we replace $h(n, k)$ by its value from **(c)**.　∎
(e3) We have,

$$\lfloor u(n, k+1)\varphi \rfloor = \lfloor \lfloor g(n,k)\varphi^2 \rfloor \varphi \rfloor = \lfloor (g(n,k) + g(n, k+1))\varphi \rfloor$$
$$\leq g(n, k+1) + g(n, k+2) + 1 = u(n, k+2) + 1,$$

where the last equality follows from **(b)**. On the other hand,

$$\lfloor (g(n,k) + g(n, k+1))\varphi \rfloor \geq g(n, k+1) + g(n, k+2) = u(n, k+2).$$

Hence by disjointness, $\lfloor u(n, k+1)\varphi \rfloor = u(n, k+2) + 1$.　∎

We recall the following special case of Lemma 2 of [5]:

Lemma I. *For integers $i > j \geq 0$ and integer $N_{i+1} \in \mathbb{Z} \geq 1$, let $R(N_{i+1}) = F_i + F_{i-2} + \cdots + F_j$, where $j = 0$ if i is even, $j = 1$ if i is odd. Then $N_{i+1} = F_{i+1} - 1$.*

This is the analog in the Fibonacci numeration system of the decimal $99 \ldots 9$.

Proof of Theorem 1 (f1). For any $n \in \mathbb{Z}_{\geq 1}$, clearly $\varphi - 1 < \Delta_g(n, 1) < \varphi + 1$, so $\Delta_g(n, 1) \in \{1, 2\} = \{F_0, F_1\}$. This shows already that S_1, S_2 split $\mathbb{Z}_{\geq 1}$. Moreover, if $\Delta_g(n, 1) = 1$ for all large n, then, since $h(n, 1)$ is increasing, we would have $g(n, 1) \cap h(n, 1) \neq \emptyset$ for infinitely many $n \in \mathbb{Z}_{\geq 1}$, contradicting the complementarity of the 2 sequences. If $\Delta_g(n, 1) = 2$ for all large n, then also $\Delta_h(n, 1) = 2$ for all large n by complementarity. But a direct computation shows that $\Delta_h(n, 1) = \Delta_g(n, 1) + 1 = 3$ for all $n \in \mathbb{Z}_{\geq 1}$, another contradiction. Thus each of S_1 and S_2 is infinite as claimed.　∎

(f2) We proceed by induction on k. Suppose that for some $k \geq 1$, $\Delta_g(n, k) = F_{k-1}$ for all $n \in S_1$, and $\Delta_g(n, k) = F_k$ for all $n \in S_2$. This holds trivially for $k = 1$ by **(f1)**. For now let's assume that $n \in S_1$. Then

$$\Delta_g(n, k+1) = \lfloor g(n+1, k)\varphi \rfloor - \lfloor g(n, k)\varphi \rfloor < g(n+1, k)\varphi - g(n, k)\varphi + 1 = F_{k-1}\varphi + 1$$

by the induction hypothesis. Also,

$$\Delta_g(n, k+1) > g(n+1, k)\varphi - g(n, k)\varphi - 1 = F_{k-1}\varphi - 1.$$

So $\lfloor F_{k-1}\varphi \rfloor \leq \Delta_g(n, k+1) \leq \lfloor F_{k-1}\varphi \rfloor + 1$. Then Lemma 2 implies that $\Delta_g(n, k) \in \{F_k - 1, F_k, F_k + 1\}$.

In the proof of **(c)** above, it was mentioned that $R(g(n+2, 2))$ ends in 01. The same thus holds for $R(g(n+1, k+1))$ and $R(g(n, k+1))$ for all $k \geq 1$, since G_{k+1} is a subsequence of G_2 for all $k \geq 1$. Therefore $R(\Delta_g(n, k+1))$ ends in 00, the same as $R(F_k)$. But $R(F_k + 1)$ ends in 01, and Lemma I implies that $R(F_k - 1)$ ends in 10, or in 01, depending on whether k is even or odd. Hence

$\Delta_g(n, k+1) = F_k$ for all $n \in S_1$. The same proof shows that $\Delta_g(n, k+1) = F_{k+1}$ for all $n \in S_2$. ∎

(f3) This follows easily for $k = 1$ by considering the size of φ. For all $k \geq 1$ it follows from **(f2)**. ∎

(f4). This follows directly from **(f1)** and **(f2)**. ∎

Note. Part of the proof of **(f4)** follows directly from **(d)**:

$$h(n+1, k+1) - g(n+1, k+2) = h(n, k+1) - g(n, k+2) = F_k.$$

Hence $\Delta_h(n, k+1) = \Delta_g(n, k+2)$. But this establishes the equality part of **(f4)** only for $k \geq 3$ and doesn't prove the membership part.

(f5). (i) Induction on k. (ii) By definition, $\Delta_h(0, k) = h(1, k) - h(0, k) = h(1, k)$. The result for $h(1, k)$ follows directly from (i) and **(c)**. ∎

(g). For $k = 1$, $h(n, 2) = \lfloor h(n, 1)\varphi \rfloor = g(h(n, 1), 1)$. If the assertion holds for any $k \geq 1$, then $h(n, k+2) = \lfloor h(n, k+1)\varphi \rfloor = \lfloor g(h(n, 1), k)\varphi \rfloor = g(h(n, 1), k+1)$. ∎

(h1). Clearly $G_2 \subset G_1$, so for every $n \in \mathbb{Z}_{\geq 1}$, $g(n, 2) = g(m, 1)$ for some $m \in \mathbb{Z}_{\geq 1}$. By **(c)**, $g(n, 2)+1 = h(n, 1) \notin G_1$. But then $g(n, 2)+2 = g(m+1, 1) \in G_1$ by **(f4)** for $k = 1$. ∎

The following is a special case of *Property* 1, Sect. 5 of [5]:

Lemma II. *The set of numbers $\{R(N) : N \in G_1\}$ ends in an even (possibly 0) number of 0s, hence the complementary set of numbers $\{R(N) : N \in H_1\}$ ends in an odd number of 0s.*

(h2). By **(h1)**, $G_2 \cup (G_2 + 2) \subseteq G_1$. Choose any $g(n, 1) \in G_1$. If $g(n, 1) \in (G_2 + 2)$, we are done. So suppose that $g(n, 1) = g(m, 2) + 2$ for no $m \in \mathbb{Z}_{\geq 1}$. By **(c)**, $g(m, 2) + 1 = h(m, 1)$ for all $m \in \mathbb{Z}_{\geq 1}$, so by disjointness, $g(n, 1) = g(m, 2) + 1$ for no $m \in \mathbb{Z}_{\geq 1}$. But then $g(n, 1) = g(m, 2)$ for some $m \in \mathbb{Z}_{\geq 1}$ by **(f4)** for $k = 1$, so $g(n, 1) \in G_2$. ∎

(h3). The following is immediately implied by **(f3)**: (a) if $g(n, 1) - 1 \notin G_1$, then $g(n, 1) - 2 \in G_1$; and, conversely, (b) if $g(n, 1) - 1 \in G_1$, then $g(n, 1) - 2 \notin G_1$. Consider case (b). Lemma II then implies that $R(g(n, 1)-1)$ ends in an even *positive* number of 0s, and $R(g(n, 1))$ ends in 01. By Lemma I, $R(g(n, 1) - 2)$ then ends in 10. We now show that $R(\lfloor g(n, 1)\varphi^2 \rfloor)$ ends in 10 for all $n \in \mathbb{Z}_{\geq 1}$.

Now $R(g(n, 1))$ ends in F_{2k-2} for some $k \in \mathbb{Z}_{\geq 1}$. By Lemma 2(ii), $\lfloor F_{2k-2}\varphi^2 \rfloor = F_{2k} - 1$, and $R(F_{2k} - 1)$ ends in 10 by Lemma I, the same as $R(g(n, 1) - 2)$ for case (b). This proves that $R(\lfloor g(n, 1)\varphi^2 \rfloor)$ ends in 10 for all $n \in \mathbb{Z}_{\geq 1}$, and the right-hand-side of **(h3)**. On the other hand, let $N \in U_2$. Then $R(N)$ ends in 10, and so $N + 1$ and $N + 2$ are both in G_1. Thus $N \in G_1 - 2$, proving the left-hand-side of **(h3)**. ∎

(h4). In the proof of **(h3)** we showed that $R(\lfloor g(n, 1)\varphi^2 \rfloor)$ ends in 10 for all $n \in \mathbb{Z}_{\geq 1}$. Since $R(h(n, 1))$ ends in an odd number of 1s for all $n \in \mathbb{Z}_{\geq 1}$ by Lemma II, it follows that $R(v(n, 2))$ ends in an odd number $N \geq 3$ of 1's. Then Lemma I implies that $R(v(n, 2) - 1)$ ends in 01. Theorem 3 of [8] states that

$R(G_2)$ is the set of all numbers whose representation ends in 01, so $(V_2 - 1) \subset G_2$.
∎

3   An Application: The Flora Game

The class of impartial take-away games appears to be partitioned into two disjoint subclasses:

- *CompGames* (composite games), and

- *PrimGames*.[1]

Informally, CompGames are games that are easy to generalize to more than one or two piles; PrimGames are those for which this seems to be very hard. A well-known representative of the former is Nim, and of the latter, Wythoff's game. Some progress in generalizing Wythoff to multiple piles was recently made. Two 3-pile games that are *extensions* rather than generalizations of Wythoff were also given recently. It appears that, largely, a game belongs to class CompGames if it decomposes into a *disjunctive* sum of subgames, such as Nim, which is the Nim-sum of its pile sizes; and it belongs to class PrimGames if it is not decomposable. Hence the names CompGames (*composite* games) and PrimGames (not decomposable – *prime*). Whereas for the former there are theories both for the impartial as well as for the partizan case, there is no general theory for the latter yet, and we believe that these "lone wolf" games should be investigated more seriously.

Here we study an extension of Wythoff to *four* piles. The efforts in defining a "right" extension, and particularly in proving the validity of the winning strategy, are considerably greater than those for *three*-pile extensions. We present four winning strategies, recursive, algebraic, arithmetic and morphic. The recursive is the easiest to describe, but seems to be hardest computationally. The algebraic depends on iterations of the floor function, the arithmetic on the Fibonacci numeration system and the morphic on a morphism. The latter three are polynomial time winning strategies.

The Flora game is a 2-player game played on 4 piles of tokens. We denote positions of Flora by (a_1, a_2, a_3, a_4) with $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4$. A position in Flora is *legal* if

$$a_1 = h(n, 1) \text{ for some } n \in \mathbb{Z}_{\geq 1} \implies a_2 \geq h(n, 2) - 2.$$

It goes without saying that every pile must contain a nonnegative number of tokens at all times. Only legal positions are permitted.

The end position is $T_0 := (0, 0, 0, 0)$. The first player unable to move (because the present position is T_0) loses; the opponent wins.

There are 3 rules of move:

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[1] It's different from the partition into MathGames and PlayGames defined in [7].

I. Any positive number of tokens from up to 3 piles may be removed.

II. From a nonzero position in which 2 piles have the same size or $a_3 - a_2 = 1$, one can move to T_0.

III. If $0 < a_1 < a_2 < a_3 < a_4$, one can remove $p > 0$ from a_3, $q > 0$ from a_4 and an arbitrary nonnegative integer from a_1 and a_2, subject to:
(i) $q = p$ if $a_4 - a_3 \notin V_2$, except for the proviso that if $a_3 - p$ is the second smallest component in the quadruple moved to, then $p \neq 5$.
(ii) $q = p + 1$ if $a_4 - a_3 \in V_2$.

Note. If the position moved to under rule III(i) is (b_1, b_2, b_3, b_4), then $a_3 - p = b_i$, $a_4 - p = b_j$ for some $1 \leq i < j \leq 4$. Then $a_4 - a_3 = b_j - b_i = t$ for some $t \in \mathbb{Z}_{\geq 1}$, and normally $t \neq p$.

3.1 Algebraic Formulation of the P-positions

The set of P-positions of a game is the set of game positions game positions from which the second (*P*revious) player can force a win.

Let

$$A_n = h(n, 1), \quad B_n = g(n, 3), \quad C_n = h(n, 2), \quad D_n = h(n, 3),$$

$$A = \cup_{n=1}^{\infty} A_n, \quad B = \cup_{n=1}^{\infty} B_n, \quad C = \cup_{n=1}^{\infty} C_n, \quad D = \cup_{n=1}^{\infty} D_n,$$

$$T_n := (A_n, B_n, C_n, D_n), \quad T = \cup_{n=0}^{\infty} T_n.$$

A prefix of T of size 19 is shown in Table 1.

Notation 4. For $n \in \mathbb{Z}_{\geq 1}$, let $\Delta_{DC}(n) := h(n, 3) - h(n, 2)$, $\Delta_{DB}(n) := h(n, 3) - g(n, 3)$, $\Delta_{DA}(n) := h(n, 3) - h(n, 1)$, $\Delta_{CB}(n) := h(n, 2) - g(n, 3)$, $\Delta_{CA}(n) := h(n, 2) - h(n, 1)$, $\Delta_{BA}(n) := g(n, 3) - h(n, 1)$, $\Delta(n) = \Delta_{DC}(n) \cup \Delta_{DB}(n) \cup \Delta_{DA}(n) \cup \Delta_{CB}(n) \cup \Delta_{CA}(n) \cup \Delta_{BA}(n)$, $\Delta = \cup_{n=1}^{\infty} \Delta(n)$.

Lemma 4. (i) $\Delta_{DC}(n) = g(n, 2)$,
(ii) $\Delta_{DB}(n) = g(n, 2) + 2$,
(iii) $\Delta_{DA}(n) = u(n, 2)$,
(iv) $\Delta_{CB}(n) = 2$,
(v) $\Delta_{CA}(n) = g(n, 1)$,
(vi) $\Delta_{BA}(n) = g(n, 1) - 2$,
(vii) $\Delta = \mathbb{Z}_{\geq 1} \setminus V_2$,
(viii) $\Delta = \cup_{n=1}^{\infty} (\Delta_{DC}(n) \cup \Delta_{DB}(n) \cup \Delta_{DA}(n))$.

Proof. (i) This is Theorem 1 **(e1)** for $k = 2$.
(ii) Theorem 1 **(e2)** for $k = 2$.
(iii) $\Delta_{DA}(n) = (h(n, 3) - h(n, 2)) + (h(n, 2) - h(n, 1)) = g(n, 2) + g(n, 1) = u(n, 2)$ by Theorem 1 **(e1)** and **(b)**.
(iv) Theorem 1 **(c)**.
(v) Theorem 1 **(e1)**.
(vi) $\Delta_{BA}(n) : (g(n, 3) - h(n, 2)) + (h(n, 2) - h(n, 1)) = g(n, 1) - 2$.

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Table 1: P-positions of Flora.

n	$h(n,1)$	$g(n,3)$	$h(n,2)$	$h(n,3)$
0	0	0	0	0
1	2	1	3	4
2	5	6	8	12
3	7	9	11	17
4	10	14	16	25
5	13	19	21	33
6	15	22	24	38
7	18	27	29	46
8	20	30	32	51
9	23	35	37	59
10	26	40	42	67
11	28	43	45	72
12	31	48	50	80
13	34	53	55	88
14	36	56	58	93
15	39	61	63	101
16	41	64	66	106
17	44	69	71	114
18	47	74	76	122

(vii) Notice that for every $n \in \mathbb{Z}_{\geq 1}$, $g(n,2) \in G_1$, $g(n,2) + 2 \in G_1$ (by Theorem 1 **(h1)**), $2 \in U_2$, $g(n,1) - 2 \in G_1 \cup U_2$ (by **(h3)**. It then follows from (iii) and (v) that $\Delta = G_1 \cup U_2$. The result follows since the sets G_1, U_2, V_2 clearly partition $\mathbb{Z}_{\geq 1}$.

(viii) Follows from (i)-(iii), (vii) and Theorem 1 **(h2)**. ■

Lemma 5. *For fixed $n \in \mathbb{Z}_{\geq 1}$, let $0 < t < g(n,2)$, $t \notin V_2$. Then there exists $0 \leq m < n$ such that $t \in \Delta_{DC}(m) \cup \Delta_{DB}(m) \cup \Delta_{DA}(m)$.*

Proof. We have $t < g(n,2) = \Delta_{DC}(n) < \Delta_{DB}(n) < \Delta_{DA}(n)$. It then follows from Lemma 4(viii), that there must be some $m < n$ for which $t \in \Delta_{DC}(m) \cup \Delta_{DB}(m) \cup \Delta_{DA}(m)$. ■

Theorem 2. *The set T constitutes the set of P-positions of the game Flora.*

Proof. To begin with we note the following facts:

- Every position (A_n, B_n, C_n, D_n) is a legal position ($n \geq 0$).

- Lemma 1 implies that each of the sequences A_n, B_n, C_n, D_n is increasing (Lemma 1).

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- A, B, C, D partition $\mathbb{Z}_{\geq 1}$ (Theorem 1 **(a)**).

It evidently suffices to prove the following two statements:

(A) Every move from any position in T results in a position outside T.

(B) For every position outside T there is a move into a position in T.

(A) Clearly there is no legal move $T_1 \to T_0$. Suppose that there are positions T_n, T_m with $m < n$, $n \geq 2$ such that there is a legal move $T_n \to T_m$. This move must be of type III, since A, B, C, D partition $\mathbb{Z}_{\geq 1}$, from which it follows easily, using Lemma 4, that $A_n < B_n < C_n < D_n$ for $n \geq 2$.

By Lemma 4(vii), $\Delta_{DC}(n) \notin V_2$, so we have to consider only move III(i). We first show that $D_n - p$ can only be D_m. It cannot be A_m, since then $C_n - p < A_m$ has no place in row m of T. Suppose $D_n - p = B_m$. Then $C_n - p = A_m$. But $\Delta_{BA}(m) < \Delta_{BA}(n) = g(n,1) - 2 < \Delta_{DC}(n) = g(n,2)$, contradicting the move rule III(i). Suppose $D_n - p = C_m$. Since $C_n - B_n = 2$ for all $n \in \mathbb{Z}_{\geq 1}$, we have $C_n - p = A_m$. But $\Delta_{CA}(m) = \Delta_{BA}(m) + 2 \leq \Delta_{BA}(n) + 1 = g(n,1) - 1 < \Delta_{DC}(n) = g(n,2)$, again contradicting move rule III(i). Thus indeed $D_n - p = D_m$.

Suppose $C_n - p = C_m$. Subtracting, $\Delta_{DC}(n) = \Delta_{DC}(m)$, so $g(n,2) = g(m,2)$ which is impossible for $m < n$ since the sequence $g(\ell,2)$ is strictly increasing. Suppose $C_n - p = B_m = C_m - 2$. Then $\Delta_{DC}(n) = \Delta_{DB}(m) = \Delta_{DC}(m) + 2$ which is possible if and only if $p = 5$ and $m = n - 1$. But this case is excluded by the proviso. Finally, suppose that $C_n - p = A_m$. Then $\Delta_{DC}(n) = \Delta_{DA}(m)$. By Lemma 4 this is equivalent to $g(n,2) = u(m,2)$. This is possible for no $m < n$ by disjointness.

(B) Let $(a_1, a_2, a_3, a_4) \notin T$, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4$. If there is equality in any of these or $a_3 - a_2 = 1$, a move of type I or II leads to T_0. So we may assume $0 < a_1 < a_2 < a_2 + 1 < a_3 < a_4$. By the complementarity of A, B, C, D, a_1 appears in precisely one component of precisely one T_n, $n \geq 1$. If $a_1 = D_n$, move $a_2 \to A_n$, $a_3 \to B_n$, $a_3 \to C_n$.

So suppose that $a_1 = C_n$.

If $a_4 \geq D_n$, move $a_2 \to A_n$, $a_3 \to B_n$, $a_4 \to D_n$. So assume $a_4 < D_n$. Let

$$t := a_4 - a_3.$$

We consider 2 cases.

(a) $t \notin V_2$, and (b) $t \in V_2$.

(a) $t \notin V_2$. We have

$$0 < t = a_4 - a_3 < D_n - a_3 < D_n - a_1 = D_n - C_n = \Delta_{DC}(n) = g(n,2).$$

By Lemma 5, there exists $m < n$ such that either (i) $t = \Delta_{DC}(m)$, or (ii) $t = \Delta_{DB}(m)$, or (iii) $t = \Delta_{DA}(m)$.

For case (i), move $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move: $a_1 = C_n > A_n > A_m$, and $a_2 > a_1 = C_n > B_n > B_m$, $a_3 > a_1 = C_n > C_m$, $a_4 = a_3 + D_m - C_m > D_m$, so this move (as well as in the remainder of this proof) is of the form III.

For case (ii), move $a_1 \to A_m$, $a_2 \to C_m$, $a_3 \to B_m$, $a_4 \to D_m$. This is a legal move: $a_1 = C_n > A_n > A_m$, $a_2 > a_1 = C_n > C_m$, $a_3 > a_1 = C_n > C_m > B_m$, $a_4 = a_3 + D_m - B_m > D_m$.

For case (iii), move $a_1 \to B_m$, $a_2 \to C_m$, $a_3 \to A_m$, $a_4 \to D_m$. This is a legal move: $a_1 = C_n > B_n > B_m$, $a_2 > a_1 = C_n > C_m$, $a_3 > a_1 = C_n > C_m > A_m$, $a_4 = a_3 + D_m - A_m > D_m$.

(b) $t \in V_2$. To remind ourselves, $t = a_4 - a_3$ and we have $a_1 = C_n$, $a_4 < D_n$. Now $t - 1 \in (V_2 - 1)$. Since $(V_2 - 1) \subset G_2$ (Theorem 1 **(h4)**), we have $t - 1 = g(m, 2)$ for suitable $m \in \mathbb{Z}_{\geq 1}$. Also $\Delta_{DC}(m) = g(m, 2)$ (Lemma 4 (i)). So we move: $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move:

- $m < n$, since $\Delta_{DC}(m) = a_4 - a_3 - 1 < D_n - a_1 = D_n - C_n = \Delta_{DC}(n)$.

- $a_1 = C_n > A_n > A_m$, $a_2 > a_1 = C_n > B_n > B_m$, $a_3 > a_1 = C_n > C_m$, $a_4 = a_3 + 1 + D_m - C_m > a_1 + D_m - C_m = C_n + D_m - C_m > D_m$.

So suppose that $a_1 = B_n$.

If $a_4 \geq D_n$, then move $a_2 \to A_n$, $a_3 \to C_n$, $a_4 \to D_n$. This is a legal move, since $a_2 > a_1 = B_n > A_n$ and

$$a_3 \geq a_2 + 1 \geq a_1 + 2 = B_n + 2 = C_n.$$

Therefore we may assume $a_4 < D_n$. The proof is similar to the above case $a_1 = C_n$. We have $a_3 \geq C_n$, and $0 < t - 1 < t = a_4 - a_3 < D_n - a_3 \leq \Delta_{DC}(n)$. Hence by Lemma 5 there is $m < n$ such that, for case (a), either (i) $t = \Delta_{DC}(m)$, or (ii) $t = \Delta_{DB}(m)$, or (iii) $t = \Delta_{DA}(m)$. For case (b) we have $t - 1 = g(m, 2)$ for some $m \in \mathbb{Z}_{\geq 1}$.

(a) $t \notin V_2$.

For case (i), move $a_2 \to A_m$, $(a_3, a_4) \to (C_m, D_m)$. This is a legal move: $a_2 > a_1 = B_n > A_n > A_m$, $a_3 \geq C_n > C_m$,

$$a_4 = a_3 + D_m - C_m \geq C_n + D_m - C_m > D_m.$$

For case (ii), move $a_1 \to A_m$, $a_2 \to C_m$, $a_3 \to B_m$, $a_4 \to D_m$. This is a legal move: $a_1 = B_n > A_n > A_m$, $a_2 \geq a_1 + 1 = B_n + 1 = C_n - 1 \geq C_m$, $a_3 > a_1 = B_n > B_m$, $a_4 = a_3 + D_m - B_m \geq C_n + D_m - B_m > D_m$.

For case (iii), move $a_1 \to B_m$, $a_2 \to C_m$, $a_3 \to A_m$, $a_4 \to D_m$. This is a legal move: $a_1 = B_n > B_m$, $a_2 \geq a_1 + 1 = B_n + 1 = C_n - 1 \geq C_m$, $a_3 > a_1 = B_n > B_m > A_m$, $a_4 = a_3 + D_m - A_m > D_m$.

(b) $t \in V_2$. We have $t = a_4 - a_3$, $a_1 = B_n$, $a_4 < D_n$. As in case (b) above, we move $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move:

- $m < n$, since $\Delta_{DC}(m) = a_4 - a_3 - 1 < D_n - a_3 - 1 \leq D_n - C_n - 1 < \Delta_{DC}(n)$.

- $a_1 = B_n > A_n > A_m$, $a_2 > a_1 = B_n > B_m$, $a_3 \geq C_n > C_m$, $a_4 = a_3 + 1 + D_m - C_m \geq C_n + 1 + D_m - C_m > D_m$.

Finally, we consider the case $a_1 = A_n = h(n, 1)$.

Since the position is legal, we have $a_2 \geq h(n-2) - 2 = B_n$. Since $a_3 - a_2 > 1$, we have $a_3 \geq B_n + 2 = C_n$. If $a_4 \geq D_n$, then at least one of the inequalities for a_2, a_3, a_4 is strict, since $(a_1, a_2, a_3, a_4) \notin T$. Then move $(a_1, a_2, a_3, a_4) \rightarrow (A_n, B_n, C_n, D_n)$. If $a_4 < D_n$ then for case (a) there is $m < n$ such that $0 < t = a_4 - a_3 < \Delta_{DC}(n)$. Hence by Lemma 5, there is $m < n$ such that either (i) $t = \Delta_{DC}(m)$, or (ii) $t = \Delta_{DB}(m)$, or (iii) $t = \Delta_{DA}(m)$. For case (b), $0 < t - 1 = g(m, 2)$ for some $m \in \mathbb{Z}_{\geq 1}$.

(a) $t \notin V_2$. For case (i) move $(a_1, a_2, a_3, a_4) \rightarrow (A_m, B_m, C_m, D_m)$. This is a legal move, since $m < n$ implies $a_1 = A_n > A_m$, $a_2 \geq B_n > B_m$, $a_3 \geq C_n > C_m$, $D_n > D_m$. For case (ii), move $a_1 \rightarrow A_m$, $a_2 \rightarrow C_m$, $a_3 \rightarrow B_m$, $a_4 \rightarrow D_m$. This is a legal move: $a_1 = A_n > A_m$, $a_2 \geq B_n = C_n - 2 > C_m$, where the strict inequality follows since $C_n - C_m \geq 3$ (Theorem 1 **(f4)** for $k = 3$). Also $a_3 \geq C_n > B_n > B_m$, $a_4 = a_3 + D_m - B_m > D_m$. For case (iii) move $a_1 \rightarrow B_m$, $a_2 \rightarrow C_m$, $a_3 \rightarrow A_m$, $a_4 \rightarrow D_m$. We have to prove the legality of this move. We begin by showing that $a_1 = A_n > B_m$. Notice that

$$
\begin{aligned}
t &= a_4 - a_3 = \Delta_{DA}(m) = u(m, 2) \quad \text{(Lemma 4)} \\
&< \Delta_{DC}(n) = g(n, 2) \quad \text{(Lemma 4)} \\
&= h(n, 1) - 1 \quad \text{(Theorem 1 (c))}.
\end{aligned}
$$

Thus $h(n, 1) > u(m, 2) + 1$. But $h(n, 1) = A_n$ and $u(m, 2) = g(m, 3) + 1$ (by Theorem 1 **(d)**) $= B_m + 1$, so indeed $A_n > B_m + 2 > B_m$. Next,

$$
\begin{aligned}
a_2 &\geq B_n = g(n, 3) = h(n, 2) - 2 \quad \text{(Theorem 1 (d))} \\
&> h(n - 1, 2) \quad \text{(Theorem 1 (f4))} = C_{n-1} \geq C_m.
\end{aligned}
$$

Also $a_3 \geq C_n > C_m > A_m$ and $a_4 = a_3 + D_m - A_m \geq C_n + D_m - A_m > D_m$.

(b) $t \in V_2$. We have $t = a_4 - a_3$, $a_1 = A_n$, $a_4 < D_n$, $t - 1 = g(m, 2) = \Delta_{DC}(m)$ for some $m \in \mathbb{Z}_{\geq 1}$. As above we move $(a_1, a_2, a_3, a_4) \rightarrow (A_m, B_m, C_m, D_m)$. This is a legal move:

- $m < n$, since $\Delta_{DC}(m) = a_4 - a_3 - 1 < D_n - C_n - 1 < \Delta_{DC}(n)$.

- $a_1 = A_n > A_m$, $a_2 \geq B_n > B_m$, $a_3 \geq C_n > C_m$, $a_4 = a_3 + 1 + D_m - C_m > C_n + D_m - C_m > D_m$. ∎

3.2 Recursive Formulation of the P-positions

Let $S \subsetneq \mathbb{Z}_{\geq 1}$ and $\overline{S} = \mathbb{Z}_{\geq 1} \setminus S$. The "Minimum EXcludant" of S is defined by

$$\text{mex } S = \min \overline{S} = \text{least positive integer not in S.}$$

In particular, the mex of the empty set is 1. (This somewhat nonstandard definition of the mex function is needed for §5.)

Let $T_0' = (0,0,0,0)$, $T_1' = (2,1,3,4)$. If $T_m' := (A_m', B_m', C_m', D_m')$ has already been defined for all $m < n$ $(n \geq 2)$, then let

$$A_n' = \mathrm{mex}\{A_i', B_i', C_i', D_i' : 0 \leq i < n\},$$

$$B_n' = \begin{cases} B_{n-1}' + 3 & \text{if } A_n' - A_{n-1}' = 2 \\ B_{n-1}' + 5 & \text{otherwise,} \end{cases}$$

$$C_n' = B_n' + 2,$$

$$D_n' = \begin{cases} D_{n-1}' + 5 & \text{if } A_n' - A_{n-1}' = 2 \\ D_{n-1}' + 8 & \text{otherwise.} \end{cases}$$

Let $T' := \cup_{n=0}^{\infty} T_n'$.

Theorem 3. *The set T' constitutes the set of P-positions of the game Flora.*

Proof. We show that for all $m \in \mathbb{Z}_{\geq 0}$, $A_m' = A_m$, $B_m' = B_m$, $C_m' = C_m$, $D_m' = D_m$. Suppose this holds for all $m < n$ $(n \geq 1)$. Let $E = \mathrm{mex}\{A_i, B_i, C_i, D_i : 0 \leq i < n\}$. The value E cannot have been assumed in any of the four sequences for $m < n$, since A, B, C, D split $\mathbb{Z}_{\geq 1}$, so $E \geq A_n$. If $E > A_n$, then A_n would never be assumed since the sequences are strictly increasing, again contradicting the complementarity of the sequences. Thus $A_n = E = A_n'$, and the other 3 equalities follow from Theorem 2 **(f)**. ■

The definition of the set T' is straightforward; it doesn't use the functions h, g used for defining T. Thus the recursive computation of T' looks easier than that of the set T. Moreover, the proof of Theorem 3 is very short, and that of Theorem 2 is long.

However, the proof of Theorem 3 leans heavily on Theorems 2 and 1. Let's examine how easy the computation of T' is. If the initial position of the game is (a_1, a_2, a_3, a_4), the input size is $\log a_1 + \log a_2 + \log a_3 + \log a_4$. The time needed to compute whether the position is a P-position or not, however, is proportional to $a_1 + a_2 + a_3 + a_4$, because the unwieldy mex function requires to scan previous entries of the sequences A_n, B_n, C_n, D_n. So the algorithm isn't all that easy; in fact, it requires exponential space (and hence exponential time)! But the decision problem whether or not (a_1, a_2, a_3, a_4) is a P-position using T is polynomial, since it requires computation of φ only up to $O(\log a_1)$ bits (see [5], §3). So the bottom line is that Theorem 3 looks easy but is hard, whereas Theorem 2 looks hard but is easy.

3.3 Arithmetic Formulation of the P-positions

For $N \in \mathbb{Z}_{\geq 1}$, let $R(N) = (d_m, \ldots, d_0)$ be the representation of N in the Fibonacci numeration system (recall Notation 3). Then $(d_m, \ldots, d_0, 0)$ is the *left shift* of $R(N)$.

Theorem 4. $R(A)$ *is the set of all representations that end in an odd number of 0-bits in the Fibonacci numeration system, $R(B)$ the set of all representations that end in* 001, *$R(C)$ the set of all representations that end in a positive even number of 0-bits, and $R(D)$ the set of all representations that end in* 101. *Moreover, for every $n \in \mathbb{Z}_{\geq 1}$, $R(C_n)$ is the left shift of $R(A_n)$.*

See Table 3.3 for an example.

Table 3.3: Representation of the P-positions in the Fibonacci numeration system.

21	13	8	5	3	2	1	A_n	n	B_n	34	21	13	8	5	3	2	1
					1	0	2	1	1								1
		1	0	0	0		5	2	6					1	0	0	1
		1	0	1	0		7	3	9				1	0	0	0	1
		1	0	0	1	0	10	4	14			1	0	0	0	0	1
	1	0	0	0	0	0	13	5	19			1	0	1	0	0	1
	1	0	0	0	1	0	15	6	22		1	0	0	0	0	0	1
	1	0	1	0	0	0	18	7	27		1	0	0	1	0	0	1
	1	0	1	0	1	0	20	8	30		1	0	1	0	0	0	1
1	0	0	0	0	1	0	23	9	35	1	0	0	0	0	0	0	1
1	0	0	1	0	0	0	26	10	40	1	0	0	0	1	0	0	1

34	21	13	8	5	3	2	1	C_n	n	D_n	55	34	21	13	8	5	3	2	1
					1	0	0	3	1	4							1	0	1
			1	0	0	0	0	8	2	12					1	0	1	0	1
			1	0	1	0	0	11	3	17				1	0	0	1	0	1
		1	0	0	1	0	0	16	4	25			1	0	0	0	1	0	1
	1	0	0	0	0	0	0	21	5	33			1	0	1	0	1	0	1
	1	0	0	0	1	0	0	24	6	38		1	0	0	0	0	1	0	1
	1	0	1	0	0	0	0	29	7	46		1	0	0	1	0	1	0	1
	1	0	1	0	1	0	0	32	8	51		1	0	1	0	0	1	0	1
1	0	0	0	0	1	0	0	37	9	59	1	0	0	0	0	0	1	0	1
1	0	0	1	0	0	0	0	42	10	67	1	0	0	0	1	0	1	0	1

Proof. The proof is similar to that of Theorem 3 of [8]. For every $m \in \mathbb{Z}_{\geq 1}$, $R(\lfloor m\varphi \rfloor)$ ends in an even number of 0-bits (including 0 0-bits), and $R(\lfloor m\varphi^2 \rfloor)$ ends in an odd number of 0-bits ([5], §4). Hence $R(A)$ is the set of all numbers that end in an odd number of 0-bits in the Fibonacci numeration system, whereas each of the other 3 representations ends in an even number of 0-bits. Now $R(C)$ is the set of all numbers that end in a positive even number of 0-bits ([8]), hence $R(B)$ and $R(D)$ each end in a 1-bit. Recall that $C_n = B_n + 2$. If

$R(B)$ would contain a number with representation ending in 101, then adding 2 to it would end in 1 (since 2+3=5 is the next Fibonacci number), contradicting the form of $R(C_n)$. Therefore $R(B)$ is the set of all numbers ending in 001. By complementarity, $R(D)$ is therefore the set of all numbers ending in 101.

Since $R(A)$ is the set of all representations ending in an odd number of 0-bits, and $R(C)$ is the set of all representations ending in a positive even number of 0-bits, the latter is the left shift of the former. Suppose that $R(C_m)$ is the left shift of $R(A_m)$ for every $m < n$. If $R(C_n)$ would not be the left shift of $R(A_n)$, then it would be assumed later on (by complementarity), contradicting the strict increase of C. ∎

This formulation of the P-positions is also easily seen to lead to a polynomial-time winning strategy.

4 The Morphic Approach

In this section we show how to construct G_k recursively by a morphism for every $k \in \mathbb{Z}_{\geq 1}$. Similar methods can be used to construct other functions defined in §2, and we also present our fourth formulation of the P-positions of the game Flora. We resort to the language of *words*. A word is any sequence over some alphabet. Here we restrict attention to binary words. A *subword* (or *factor*) of a word is any contiguous subset of symbols. Thus 00101 and 101100 are subwords of the word $w = 010010110001$. The *length* of a (sub)word is the number of its symbols, counting multiplicities. The lengths of the above 2 subwords and of w are 5, 6 and 12 respectively. The length of any (sub)word w is denoted by $|w|$.

4.1 Morphism for G_k

Define the morphism $1 \to 10$, $0 \to 1$. Its fixed point is the word: $F = 1011010110110\ldots$, also known as the Fibonacci word. For $k \geq 1$, the characteristic function χ_k of G_k is defined by

$$\chi_k(m) = \begin{cases} 1 \text{ if } \exists\ n \text{ s.t. } g(n,k) = m \\ 0 \text{ otherwise.} \end{cases}$$

Definition 1. Given a binary word W. A *run* of 0s is any (possibly empty) subword of W consisting solely of 0s, flanked on the left and right by a 1-bit. A *block* is any subword consisting of a 1-bit followed by a run of 0s.

Theorem 5. *For every $k \geq 1$, the "substitution morphism" for producing the characteristic function χ_k of G_k, beginning with $10^{F_{k-1}-1}$, is:*

$$10^{F_{k-1}-1} \to 10^{F_k-1}, \quad 0^{F_{k-2}} \to 10^{F_{k-1}-1}.$$

Proof. The substitution morphism is well-defined. Indeed, the initial block of length F_{k-1} is mapped into a block B_1 of length F_k. In the second iteration, the prefix of length F_{k-1} of B_1 is again mapped into B_1. The remaining abutting

suffix of B_1 consists of F_{k-2} 0s, so it is mapped into a block B_2 of length F_{k-1}. In the third iteration, B_1 and B_2 are generated again, and then the block B_2 of length F_{k-1} generates a block B_1. Thus for all subsequent iterations only blocks of the form B_1 and B_2 are generated, and there is never any parsing conflict.

Notice that for G_1, the "substitution morphism" is simply the well-known morphism $1 \rightarrow 10$, $0 \rightarrow 1$, which produces F. Moreover, $\chi_1 = F$. See e.g., [1], ch. 9 and [11].

Since $\chi_1 = 1011010110110\ldots$, Theorem 1($\mathbf{f}$) implies that $\chi_2 = 10010100100$ $1010010100\ldots$, where we inserted into χ_1 $F_1 - F_0 = 1$ zero to each run of $F_0 - 1 = 0$ zeros (i.e., one 0 between every consecutive 1s), and $F_2 - F_1 = 1$ zero to each run of $F_1 - 1 = 1$ zeros. Doing this yields distances between consecutive 1s in χ_2 of F_1 and F_2, precisely at the locations where the distances between consecutive 1s of χ_1 are F_0 and F_1 respectively. Similarly, $\chi_3 = 100001001000010000100100000100010000\ldots$, where we inserted into χ_2 $F_2 - F_1 = 1$ zero to each run of $F_1 - 1 = 1$ zero, and $F_3 - F_2 = 2$ zeros to each run of $F_2 - 1 = 2$ zeros.

In general, for producing χ_{k+1} from χ_k, we add to χ_k $F_k - F_{k-1} = F_{k-2}$ zeros to each run of $F_{k-1} - 1$ zeros and $F_{k+1} - F_k = F_{k-1}$ zeros to each run of F_k zeros. This yields blocks of sizes F_k and F_{k+1} respectively, at the locations specified by Theorem 1($\mathbf{f}$).

Assume inductively that the substitution morphism

$$10^{F_{k-1}-1} \rightarrow 10^{F_k-1}, \quad 0^{F_{k-2}} \rightarrow 10^{F_{k-1}-1}$$

produces χ_k, so it generates distances between consecutive 1s of F_{k-1} and F_k at the locations specified by Theorem 1($\mathbf{f}$). Then the substitution morphism

$$10^{F_k-1} \rightarrow 10^{F_{k+1}-1}, \quad 0^{F_{k-1}} \rightarrow 10^{F_k-1}$$

produces χ_{k+1}, since it adds $F_{k+1} - F_k$ to the F_k 0s of the long 0-runs of χ_k, and $F_k - F_{k-1}$ 0s to the F_{k-1} short 0-runs of χ_k. $\blacksquare$

4.2 Morphic Formulation of of the P-positions

Denote terms of A_n, B_n, C_n, D_n by a, b, c d respectively.

Theorem 6. *The morphism*

$$bac \rightarrow bacda, \qquad da \rightarrow bac,$$

beginning with bac, generates the characteristic function of the P-positions of the Flora game.

Proof. The proof is rather similar to that of Theorem 5, and is therefore omitted. $\blacksquare$

This theorem also leads to a polynomial-time winning strategy, since induction shows that for every $k \in \mathbb{Z}_{\geq 1}$, the k-th application of the morphism generates a word of length F_{k+2}.

Notice that if we replace *bac* by 1 and *da* by 0, we get back our old morphic friend $1 \to 10$, $0 \to 1$.

4.3 Characterization of the Sequences G, H, by the Fibonacci Numeration System

We know from Lemma II and §3.3, that $R(g(n,1))$ ends in an even number of 0s, $R(h(n,1))$ in an odd number of 0s, $R(h(n,2))$ in an even positive number of 0s, $R(h(n,3))$ in 101, and $R(g(n,3))$ in 10^s1, $s \geq 2$. What's the general pattern?

Theorem 7. (i) *$R(G_1)$ is the set of all representations that end in an even number of 0s, $R(H_1)$ is the set of all representations that end in an odd number of 0s, $R(G_2)$ is the set of all representations that end in a 1-bit, and $R(H_2)$ is the set of all representations that end in an even positive number of 0s.*
(ii) *For every $k \in \mathbb{Z}_{\geq 3}$ and all $n \in \mathbb{Z}_{\geq 1}$, $R(G_k)$ is the set of all representations that end in the word 10^s1 for all $s \geq k-1$, and $R(H_k)$ is the set of all representations that end in the word $10^{k-2}1$ (left 1-bit in position $k-1$).*
(iii) *For every $n \in \mathbb{Z}_{\geq 1}$, $R(h(n,2))$ is the left shift of $R(h(n,1))$.*

Proof. Items (i) and (iii) are already known from Theorem 4 and Lemma II, and are included here only for the sake of completeness. We only have to point out the statement about $R(G_2)$, which follows from the fact that G_2, H_2, H_1 split the positive integers (see also [8], Theorem 3).

(ii) Induction on k. The base case $k = 3$ was proved in Theorem 4. For $k \geq 3$, suppose that we already proved that $R(G_k)$ is the set of all representations that end in 10^s1 for all $s \geq k-1$, and $R(H_k)$ is the set of all representations that end in $10^{k-2}1$. It clearly remains only to show that $R(G_{k+1})$ is the set of all representations that end in 10^s1 for all $s \geq k$, and $R(H_{k+1})$ is the set of all representations that end in the word $10^{k-1}1$. Recall Theorem 1 **(c)**: $h(n,k) = g(n,k+1) + F_{k-1}$. If $R(G_{k+1})$ would contain a number, say $g(n,k+1)$, with representation ending in $10^{k-1}1$ (with leftmost 1-bit in position k), then adding F_{k-1} to it would result in a word with representation ending in 0^k1 because $F_{k-1} + F_k = F_{k+1}$ is the *next* Fibonacci number. But then $R(h(n,k))$ would end in 10^s1 for some $s \geq k$, contradicting the induction hypothesis. Thus $R(H_{k+1})$ is the set of all representations that end in the word $10^{k-1}1$. Since $G_{k+1}, H_{k+1}, H_k, \ldots, H_2, H_1$ split the integers, $R(H_{k+1})$ is the set of all representations that end in the word $10^{k-1}1$. ∎

Notes. (1) The general pattern of the representation of the suffixes of H_k for $k \geq 3$ is quite different from that of H_1 and H_2, and both of these are different from each other. The same holds for G_k, $k \geq 3$ and G_1 and G_2. Therefore the induction proof could not have begun with $k = 1$ or 2.
(2) The statement in (i) about $R(G_1)$ and $R(H_1)$ is Theorem 9.1.15 (see also Corollary 9.1.14) in [1], credited there to [10]. (It is also Lemma II.) The proof method of [1] follows [3].

5 Infinite Complementary Arrays

The doubly infinite *Stolarsky Array* A with entries $A(i,j)$, $i,j \geq 1$ [18], is defined as follows: For every $m \geq 1$, $A(m,1) = \text{mex}\{A(i,j) : i < m, j \geq 1\}$, $A(m,2) = \lfloor A(m,1)\varphi + 1/2 \rfloor$, and for all $i \geq 1$, $j \geq 3$, $A(i,j)$, $A(i,j) = A(i,j-1) + A(i,j-2)$. Then every positive integer appears precisely once in A. A beginning portion is exhibited in Table 2. Many variations, interspersions and dispersions have since been given, see e.g., [14], [15]. All are doubly infinite, $\lim_{j \to \infty}(A(i,j+1) - A(i,j)) = \infty$ for every $i \geq 1$, and every positive integer appears precisely once in A.

Table 2: The Stolarsky array

1	2	3	5	8	13	...
4	6	10	16	26	42	...
7	11	18	29	47	76	...
9	15	24	39	63	102	...
⋮	⋮	⋮	⋮	⋮	⋮	

For every $k \geq 1$, define the Flora-Array L_K with the $k+1$ rows $H_1, H_2, \ldots, H_k, G_k$. This array has a different character. By Theorem 1(a), this *singly*-infinite array also has the property that every positive integer appears precisely once. Moreover, $A(i,j+1) - A(i,j) \in \{F_i, F_{i+1}\}$ is bounded for every fixed i and all $j \geq 1$, but $\lim_{i \to \infty}(A(i,j+1) - A(i,j)) = \infty$. Table 3 depicts the case $k = 6$. The two bottom lines, below the horizontal line, illustrate the fact that G_7, H_7 split G_6, so replacing G_6 by H_7 and G_7 constitutes Floor$_7$.

Table 3: A complementary Flora array L_6 with 7 rows

n	1	2	3	4	5	6	7	8	9	10	11	12	13
H_1	2	5	7	10	13	15	18	20	23	26	28	31	34
H_2	3	8	11	16	21	24	29	32	37	42	45	50	55
H_3	4	12	17	25	33	38	46	51	59	67	72	80	88
H_4	6	19	27	40	53	61	74	82	95	108	116	129	142
H_5	9	30	43	64	85	98	119	132	153	174	187	208	229
H_6	14	48	69	103	137	158	192	213	247	281	302	336	370
G_6	1	22	35	56	77	90	111	124	145	166	179	200	221
H_7	22	77	111	166	221	255	310	344	⋯				
G_7	1	35	56	90	124	145	179	200	234	268	289	323	⋯

6 Epilogue

We have generated sequences consisting of nested arbitrary applications of the floor function to φ and φ^2, established many identities and relationships involving them and then applied them to formulate an algebraic winning strategy to the game Flora. We also presented recursive, arithmetic and morphic formulations of the winning strategy. In addition, we characterized the main sequences by means of the Fibonacci numeration system, and generated infinite complementary arrays of the sequences.

Can some of the relationships of the iterated floor functions be generalized to reals other than φ and φ^2? As a first step, in the definition of $g(n, k)$ and $h(n, k)$, we could interchange φ with φ^2, studying the ensuing sequences and the games implied by them. Specifically, define $g'(n, 1) = \lfloor n\varphi^2 \rfloor$, $h'(n, k) = \lfloor n\varphi \rfloor$, and for $k \geq 2$, $g'(n, k) = \lfloor g'(n, k-1)\varphi^2 \rfloor$, $h'(n, k) = \lfloor g'(n, k-1)\varphi^2 \rfloor$. If we define G'_i, H'_i in the obvious way, it's straightforward to see that then the first item of Theorem 1, is preserved, namely $G'_k, H'_k, H'_{k-1}, \ldots, H'_2, H'_1$ partition $\mathbb{Z}_{\geq 1}$. What games can be spawned from this partition?

More generally, it would be well to investigate which of the above results hold for which classes of positive reals beyond φ. For example, Lemma 1 and Theorem 1(**a**) clearly hold if we replace φ by any irrational $\alpha \in (1, 2)$ and φ^2 by $\beta = \alpha/(\alpha - 1)$. Perhaps large parts of Theorem 1 can be generalized for the case where $\alpha = (2 - t + \sqrt{t^2 + 4})/2$, $\beta = \alpha + t$, where t is any given positive integer, since then the simple continued fraction of α is $[1, t, t, t, \ldots]$, so the numeration system arguments used in the proof of (**c**) and (**d**) of Theorem 1 carry over in a simple way. What games are induced by these relationships?

The notion of arbitrary iterations of the floor function appeared in [17] and [13] and perhaps in other papers I was unable to locate. In the former, the iterations are with *rational* numbers whose sizes depend on the iteration depth; in the latter, the aim is to represent the positive integers in the form of iterated floor functions involving φ and φ^2.

The Raleigh game [8] is an extension — not generalization — of Wythoff's game. Flora is an extension of Raleigh. Although Flora appears not to be decomposable into sums of more elementary games, we were able to formulate for it three polynomial-time winning strategies. The one based on the Fibonacci numeration system is of particular interest. It demonstrates once again that numeration systems can make strategies of games in PrimGame efficient, similarly to appropriate data structures (see [16]).

We can also define a 5-pile extension of Flora, but in the sequence of games with increasing number of piles, both the definition of the games and the validity proof of their strategies seem to become more difficult. For example, whereas the union of the differences Δ between the 3 columns of the P-positions of Raleigh covers all of $\mathbb{Z}_{\geq 1}$, the same union for the four columns of the P-positions of Flora leaves out V_2. But perhaps a pattern for these games will emerge. This possibility may not be so far-fetched, since, as we saw, e.g., in §4.3, the general behavior begins only with $k = 3$ (corresponding to a game with 4 piles).

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