Infinite cyclic impartial games

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Abstract

We define the family of *locally path-bounded* digraphs, which is a class of infinite digraphs, and show that on this class it is relatively easy to compute an optimal strategy (winning or nonlosing); and realize a win, when possible, in a finite number of moves. This is done by proving that the Generalized Sprague-Grundy function exists uniquely and has finite values on this class.
Keywords: infinite cyclic games, locally path-bounded digraphs, generalized Sprague-Grundy function

1 Introduction

We are concerned with *combinatorial games*, which, for our purposes here, comprise 2-player games with perfect information, no chance moves and outcome restricted to (lose, win), (draw, draw) for the two players. A draw position is a position in the game such that no win is possible from it, but there exists a next move which guarantees, for the player making it, not to lose. You *win* a game by making a last move in it. A game is *impartial* if for every position in it, both players have the same set of next moves; otherwise it's *partizan*. Nim is impartial, chess partizan. A game is *cyclic* if it contains cycles (the possibility of returning to the same position), or loops (pass-positions). These notions, slightly changed here, can be found in [1]. It is clear that a necessary (yet not sufficient) condition for the existence of draw positions is that the game be cyclic.

For Partizan cyclic games, see [3], [12], [4], [10], [5]; finite impartial cyclic games are discussed only briefly in [1], [2]. Particular finite impartial cyclic games are analyzed in [9], [7]. Infinite impartial games are treated briefly at the end of [14], where both the "generalized Sprague-Grundy function" γ, defined below, and its associated "counter function" were permitted to be transfinite ordinals.

Our purpose here is to define a certain class of infinite digraphs on which γ assumes only finite values, but the counter function may contain transfinite ordinal values. The motivation for doing this is based, in part, on the following considerations. It is easier to compute with finite than with transfinite ordinals. For just determining who of the two players can win (or that both can at most draw) in a *sum* of games, the "generalized Nim-sum" of a finite set of γ-values seems to be required (§4). The generalized Nim-sum is based on the binary expansion of ordinals. It's easy to see that every ordinal, finite or transfinite, has a unique expansion as a *finite* sum of powers of ordinals (based on the greedy algorithm and the fact that the ordinals are well-ordered — see [13, XIV, §19]). For example, $\omega = 2^\omega$. We do not wish to enter here into the question of the computational complexity of computing with transfinite ordinals. But it seems possible that it's easier to compare the size of ordinals with each other, which suffices for counter function values, than to compute and work with their binary expansions, as needed for the γ-values. The counter function helps for consummating a win, and this can be done, in a finite number of moves even if the counter function value is a transfinite ordinal, since the ordinals are well-ordered. Moreover, often the structure of the digraph is such that the γ-function itself suffices to provide a winning strategy, without the need of an additional counter function (§4).

The connection between games and digraphs is simple: with any impartial game Γ we associate a digraph $G = (V, E)$ where V is the set of positions of Γ and $(a, b) \in E$ if and only if there is a move from position a to position b. It is called the *game-graph* of Γ. We identify games with their corresponding game-graphs, game positions with digraph vertices and game moves with digraph edges, using them interchangeably. It is thus natural to define a *cyclic* digraph as a digraph, finite or infinite, which may contain cycles or loops.

In §2 we provide basic tools needed for the statement and proof of the result (Theorem 1), and §3 contains the proof. An example demonstrating Theorem 1 is given in the final §4.

2 Preliminaries

The subset of nonnegative integers is denoted by $\mathbb{Z}^0$, and the subset of positive integers by $\mathbb{Z}^+$.

Given a digraph $G = (V, E)$. For any vertex $u \in V$, the set of *followers* of u is $F(u) = \{v \in V : (u, v) \in E\}$. A vertex u with $F(u) = \emptyset$ is a *leaf*. The set of *predecessors* of u is $F^{-1}(u) = \{w \in V : (w, u) \in E\}$. A *walk* in G is any sequence of vertices $u_1, u_2, \ldots$, not necessarily distinct, such that $(u_i, u_{i+1}) \in E$, i.e., $u_{i+1} \in F(u_i)$ $(i \in \mathbb{Z}^+)$. Edges may be repeated. A *path* is a walk with all vertices distinct. In particular, there's no repeated edge in a path. The *length* of a path is the number of its edges. If every path in G has finite length, then G is called *path-finite*. If there exists $b \in \mathbb{Z}^0$ such that every path in G has length

$\leq b$, then G is *path-bounded*.

Definition 1. A cyclic digraph is *locally path-bounded* if for every vertex u_i there is a bound $b_i(u_i) = b_i \in \mathbb{Z}^0$ such that the length of every (directed) path emanating from u_i doesn't exceed b_i. The integer b_i is the *local path bound* of u_i.

Note that every path-bounded digraph is locally path-bounded, and every locally path-bounded digraph is path-finite. But neither of the two inverse relationships needs to hold. Our main result is concerned with locally path-bounded digraphs.

Given a digraph $G = (V, E)$. The *Generalized Sprague-Grundy function*, also called *γ-function*, is a mapping $\gamma: V \to \mathbb{Z}^0 \cup \{\infty\}$, where the symbol ∞ indicates a value larger than any natural number. If $\gamma(u) = \infty$, we say that $\gamma(u)$ is infinite. We wish to define γ also on certain subsets of vertices. Specifically: $\gamma\big(F(u)\big) = \big\{\gamma(v) < \infty : v \in F(u)\big\}$. If $\gamma(u) = \infty$ and $\gamma\big(F(u)\big) = K$, we also write $\gamma(u) = \infty(K)$. Next we define equality of $\gamma(u)$ and $\gamma(v)$: if $\gamma(u) = k$ and $\gamma(v) = \ell$ then $\gamma(u) = \gamma(v)$ if one of the following holds: (a) $k = \ell < \infty$; (b) $k = \infty(K)$, $\ell = \infty(L)$ and $K = L$. We also use the notations

$$V^f = \big\{u \in V : \gamma(u) < \infty\big\} , \qquad V^\infty = V \setminus V^f ,$$

$$\gamma'(u) = \operatorname{mex} \gamma\big(F(u)\big) = \operatorname{mex}\{\gamma(v) < \infty : v \in F(u)\} , \tag{1}$$

where for any finite subset $S \subset \mathbb{Z}^0$, the *M*inimum *EX*cluded value mex is defined by

$$\operatorname{mex} S = \min(\mathbb{Z}^0 \setminus S) = \text{minimum term in } \mathbb{Z}^0 \text{ not in } S .$$

We need some device to tell the winner where to go when we use the γ-function. For example, suppose that there is a token on vertex u (Fig. 1). It turns out that it's best for the player moving now to go to a position with γ-value 0. There are two such values: one (the leaf) is an immediate win, and the other (v) is only a nonlosing move. This digraph may be embedded in a large digraph where it's not clear which option leads to a win. The device which overcomes this problem is a counter function, as used in the following definition. For realizing an optimal strategy, we will normally select a follower of least counter function value with specified γ-value. The counter function also enables us to prove assertions by induction.

Definition 2. Given a cyclic digraph $G = (V, E)$. A function $\gamma: V \to \mathbb{Z}^0 \cup \{\infty\}$ is a *γ-function* with *counter function* $c: V^f \to J$, where J is any infinite well-ordered set, if the following three conditions hold:

A. If $\gamma(u) < \infty$, then $\gamma(u) = \gamma'(u)$.

B. If there exists $v \in F(u)$ with $\gamma(v) > \gamma(u)$, then there exists $w \in F(v)$ satisfying $\gamma(w) = \gamma(u)$ and $c(w) < c(u)$.

C. If $\gamma(u) = \infty$, then there is $v \in F(u)$ with $\gamma(v) = \infty(K)$ such that $\gamma'(u) \notin K$.

Remarks.

- In **B** we have necessarily $u \in V^f$; and we may have $\gamma(v) = \infty$ as in **C**.
- To make condition **C** more accessible, we state it also in the following equivalent form:

  > **C'**. If for every $v \in F(u)$ with $\gamma(v) = \infty$ there is $w \in F(v)$ with $\gamma(w) = \gamma'(u)$, then $\gamma(u) < \infty$.

- If condition **C'** is satisfied, then $\gamma(u) < \infty$, and so by **A**, $\gamma(w) = \gamma'(u) = \gamma(u)$.
- To keep the notation simple, we write $\infty(0)$, $\infty(1)$, $\infty(0,1)$ etc., for $\infty(\{0\})$, $\infty(\{1\})$, $\infty(\{0,1\})$, etc.

The γ-function was first defined in [14]. It was found independently in [8]. The simplified version given above, and two other versions, appear in [11]. Since this function is not well-known, we repeated its definition above. The γ-function exists uniquely on any finite cyclic digraph, but its associated counter function exists nonuniquely.

We say that a digraph G has a γ-function, if γ exists on G with values restricted to $\mathbb{Z}^0 \cup \{\infty\}$, i.e., no transfinite ordinal values.

We are now ready to state our main result.

Theorem 1. *Every locally path-bounded digraph $G = (V, E)$ has a unique γ-function with an associated counter function; and for every $u \in V^f$, $\gamma(u)$ doesn't exceed the length of a longest path emanating from u. The statement doesn't hold in general for path-finite digraphs.*

3 The Proof

We wish to examine some properties of path-finite and locally path-bounded digraphs. To begin with, is it clear that for a path-finite digraph, if $v \in F(u)$, then every path emanating from v is not longer than any path emanating from u?

Perhaps it is clear, but it's also wrong: in a path-finite graph, every path originating at some vertex u and continuing to its ultimate end, terminates at a vertex v, where v is either a leaf or a predecessor of some w on the path. Thus a path of minimum length emanating from u in Fig. 1 terminates at the leaf, whereas a path of maximum length beginning at u terminates at y. It has length 3. But a maximal-length path emanating from $v \in F(u)$ clearly has length 4.

If a digraph $G = (V, E)$, possibly with infinite paths, has no leaf, then the label ∞ on all the vertices is evidently a γ-function: **A** and **B** are satisfied vacuously, and **C** is satisfied with $\gamma'(u) = 0$ for all $u \in V$. If G has a leaf, then some of the vertices have a γ-function, such as the leaf and its predecessors, but possibly γ doesn't exist on some of the vertices. For the case where γ exists on a subset $V' \subseteq V$, we define $\gamma'(u) = \text{mex}\{\gamma(F(u)) : F(u) \subseteq V'\}$. Since $F(u)$ may, nevertheless, be infinite for any vertex u in a locally path-bounded digraph, it

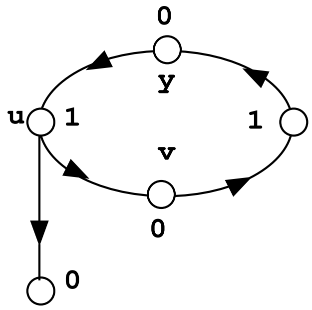

Figure 1: The numbers are γ-values.

is not clear a priori that $\gamma'(u)$ exists. The following lemma takes care of this point.

Lemma 1. *Let u be any vertex with local path bound b in a locally path-bounded digraph $G = (V, E)$. Then $\gamma'(u)$ exists (i.e., it is a nonnegative integer), and in fact, $\gamma'(u) \le b$.*

Proof. We consider two cases.

(i) Suppose that u has finite γ-value m. Then $\gamma'(u)$ exists, and in fact, $\gamma'(u) = \gamma(u) = m$ by **A**. Moreover, there exists $u_1 \in F(u)$ with $\gamma(u_1) = m - 1$, there exists $u_2 \in F(u_1)$ with $\gamma(u_2) = m - 2, \ldots$, there exists $u_m \in F(u_{m-1})$ with $\gamma(u_m) = 0$. Then $u, u_1, \ldots, u_m$ is a path of length m, so $m \le b$. (The path may continue beyond u_m, but in any case $\gamma'(u) = m \le b$.)

(ii) Suppose that u has either no γ-value or value ∞. It suffices to show that if $v \in F(u) \cap V^f$, then $\gamma(v) < b$. Indeed, $|F(u)|$ may be infinite, and $F(u)$ may contain vertices with no γ-value. But if $\gamma(v) < b$ for all $v \in F(u) \cap V^f$, then clearly $\gamma'(u)$ exists and $\gamma'(u) \le b$. Note that we cannot use the argument of case (i) directly on v, since a path from v may be longer than a path from u, as we just saw. So suppose there is $v_0 \in F(u) \cap V^f$ with $\gamma(v_0) = n \ge b$. As in case (i), there is a path $v_0, v_1, \ldots, v_n$ of length n with $\gamma(v_i) = n - i$ ($i \in \{0, \ldots, n\}$). Then $u, v_0, v_1, \ldots, v_n$ is a walk of length $n + 1 > b$ emanating from u. Hence it cannot be a path. But $v_i \ne v_j$ for all $i \ne j$, since $\gamma(v_i) \ne \gamma(v_j)$. Hence $v_j = u$ for some $j \in \{0, \ldots, n\}$. The contradiction is that v_j does and u doesn't have a finite γ-value. Thus $\gamma(v) < b$ for all $v \in F(u) \cap V^f$, hence $\gamma'(u)$ exists, and in fact, $\gamma'(u) \le b$. ∎

Proof of Theorem 1. Let $V' \subseteq V$ be a maximal subset of vertices on which γ exists, together with an associated counter function c, subject to the following additions to **B** and **C** of Definition 2:

$$\text{If } \gamma(u) < \infty \text{ and there is } v \in F(u) \cap V_\nu\,, \quad \text{then there is } w \in F(v) \text{ with } \gamma(w) = \gamma(u),\ c(w) < c(u)\,, \tag{2}$$

$$\text{if } \gamma(u) = \infty, \text{ then there is } v \in F(u) \text{ with } \gamma(v) = \infty \quad \text{such that } w \in F(v) \cap V_\nu \Longrightarrow \gamma'(w) \ne \gamma'(u)\,, \tag{3}$$

where $V_\nu = V \setminus V'$. (In (3) we have $\gamma'(w) \neq \gamma'(u)$, instead of $w \in F(v)$ and $\gamma(w) \neq \gamma'(u)$ in **C**.) In addition we require:

$$\text{If } \gamma(u) = \infty \text{ with } \gamma'(u) = l, \text{ then } \gamma'(v) \geq l \text{ for all } v \in V_\nu \ . \qquad (4)$$

The subset V' is maximal in the sense that adjoining any $u \in V_\nu$ into V' violates either Definition 2, or (2) or (3) or (4). If $V_\nu \neq \emptyset$, let $u \in V_\nu$. By Lemma 1, $\gamma'(u) = k$ exists for some $k \in \mathbb{Z}^0$. It follows that there is a minimum value $m = \min\{k \in \mathbb{Z}^0 : u \in V_\nu,\ \gamma'(u) = k\}$. Let $K = \{u \in V_\nu : \gamma'(u) = m\}$. Then $V_\nu \neq \emptyset \Longrightarrow K \neq \emptyset$. We consider four cases.

Case 1. For every $u \in K$ we have $m \in \gamma'(F(u))$, where, consistent with (1),

$$\begin{aligned}\gamma'(F(u)) = \{\gamma'(v) : v \in F(u)\} &= \{\operatorname{mex} \gamma(F(v)) : v \in F(u)\} \\ &= \{\operatorname{mex}\{\gamma(w) < \infty : w \in F(v), v \in F(u)\}\} \ .\end{aligned}$$

Note that $u \in K$, $v \in F(u) \Longrightarrow \gamma(v) \neq m$ by the definition of mex, so $v \in F(u)$, $\gamma'(v) = m \Longrightarrow v \in V_\nu$, in fact, $v \in K$. Thus putting $\gamma(u) = \infty$ for all $u \in K$ satisfies **C**, and is also consistent with (3); and with (4) by the minimality of m. Furthermore, it doesn't violate **A**, and is consistent with **B** by (2). This contradicts the maximality of V'.

We may thus assume henceforth that there exists $u \in K$ such that

$$m \notin \gamma'(F(u)) \ . \qquad (5)$$

Case 2. There exist $u \in K$ and $v \in F(u)$ with $\gamma(v) = \infty$, such that for every $w \in F(v)$, either $\gamma(w) \neq m$, or $w \in V_\nu$ with $\gamma'(w) \neq m$. Putting $\gamma(u) = \infty$ is clearly consistent with **C**, and (3); and it doesn't violate **A**. In view of (2), also **B** is satisfied. This contradicts the maximality of V'. So we may assume that

$$\begin{aligned}\forall u \in K \text{ and } \forall v \in\ & F(u) \text{ with } \gamma(v) = \infty \ , \\ & \exists w \in F(v) \ (\text{with } \gamma(w) = m \text{ or } w \in V_\nu \text{ with } \gamma'(w) = m) \ .\end{aligned}$$

We subdivide this into the following two cases:

$$\begin{aligned}\exists u \in K \text{ such that } \forall v \in F(u) \text{ with }\ & \gamma(v) = \infty \ , \\ & \exists w \in F(v) \text{ with } \gamma(w) = m \ , \qquad (6)\end{aligned}$$

or

$$\begin{aligned}\forall u \in K \text{ and } \forall v \in\ & F(u) \text{ with } \gamma(v) = \infty \ , \\ & \exists \text{ no } w \in \cap V', \text{ but } \exists w \in F(v) \cap V_\nu \text{ with } \gamma'(w) = m \ . \ (7)\end{aligned}$$

Case 3. (6) holds. We repeat that for any $u \in K$, since $\gamma'(u) = m$, u has no follower with γ-value m. Suppose that there exists $y \in F^{-1}(u)$ with $\gamma(y) = m$. Then by (2), there exists $v \in F(u)$ with $\gamma(v) = m$, contradicting $\gamma'(u) = m$. Thus putting $\gamma(u) = m$ is consistent with **A**. It is also consistent with (3):

putting $\gamma(u) = m$ could presumably increase $\gamma'(y)$ for some $y \in F^{-1}(u) \cap V_\nu$, and thus upset (3) for the value $\gamma(z) = \infty$ of some grandparent $z = F^{-1}(F^{-1}(y))$ of y. Now by (4), $\gamma'(u) \geq \gamma'(z)$. If indeed $\gamma'(y)$ increased, then for the new value we have $\gamma'(y) > \gamma'(u)$, so $\gamma'(y) > \gamma'(z)$ and $\gamma(z) = \infty$ remains unaffected. Consistency with **C** thus follows from (3) which becomes **C** when u is labeled m. Since $y \in F^{-1}(u) \Longrightarrow \gamma(y) \neq m$, as we saw at the beginning of this case, the potential adverse effect on any grandparent z of y considered above, cannot happen.

We now show that also **B** holds. Suppose first that $F(u) \subseteq V'$. For every $v \in F(u)$ for which $\gamma(v) > m$, there exists $w \in F(v)$ with $\gamma(w) = m$. This follows from **A** if $\gamma(v) < \infty$, and from (6) if $\gamma(v) = \infty$. It remains to define $c(u)$ sufficiently large so that $c(w) < c(u)$. This will be done below.

In view of the minimality of m and by (5), the second possibility is that for every $v \in F(u)$ for which $v \in V_\nu$, we have $\gamma'(v) > m$. For every such v there exists $w \in F(v)$ with $\gamma(w) = m$ by the definition of mex. Again we have to define $c(u)$ sufficiently large to satisfy $c(w) < c(u)$.

Let $S = \{v \in F(u) : \gamma(v) > m\} \cup \{v \in F(u) : v \in V_\nu,\ \gamma'(v) > m\}$. We have just seen that for every $v \in S$ there is $w \in F(v)$ with $\gamma(w) = m$. Put $T = \{w \in F(v) : v \in S,\ \gamma(w) = m\}$. Let $c(u)$ be the smallest ordinal $> c(w)$ for all $w \in T$. Then also **B** is satisfied. This contradicts the presumed maximality of V'.

Note that the case $F(u) \subseteq V \setminus V^\infty$ satisfies (6) vacuously, and so is also included in the present case.

Case 4. (7) holds. If (7) holds nonvacuously, then as in Case 1, putting $\gamma(u) = \infty$ for all $u \in K$ is consistent with **C**, (3) and the other conditions. This contradicts again the maximality of V'. Hence $K = \emptyset$, and so also $V_\nu = \emptyset$.

Whenever γ exists on a digraph, finite or infinite, it exists there uniquely. See [6]. Finally, if b is the local bound of $u \in V$, then $\gamma'(u) \leq b$ by Lemma 2. Hence if $u \in V^f$, then $\gamma(u) \leq b$ by **A**.

For proving the last statement of the theorem, consider the digraph G which consists of a vertex u, and $F(u) = \{u_0, u_1, \ldots\}$, where, for all $i \in \mathbb{Z}^0$, u_i is the top vertex of a Nim-heap of size i, so $\gamma(u_i) = i$ $(i \geq 0)$. It is clear that G is path-finite but not locally path-bounded. Also $\gamma(F(u)) = \{0, 1, \ldots\}$, so $\gamma(u)$ cannot assume any finite value. In fact, $\gamma(u) = \omega$, where ω is the smallest transfinite ordinal bigger than all the natural numbers. If, however, $F(u_i) = u_{i-1}$, and $F(u) = \{u_0, u_1, \ldots\}$ as above, then again G is path-finite but not locally path-bounded, yet $\gamma(u) = 2$. ■

4 An Example

We specify below a locally path-bounded digraph $G = (V, E)$ on some of whose vertices we place a finite number of tokens. A move consists of selecting a token and moving it to a follower. Multiple occupancy of vertices is permitted. The player first unable to move loses, and the opponent wins. If there is no last move, the outcome is a draw.

For any $r \in \mathbb{Z}^0$, a *Nim-heap* of size r is a digraph with vertices $u_0, \ldots, u_r$ and edges (u_j, u_i) for all $0 \le i < j \le r$. In depicting G (Fig. 2), we use the convention that bold lines and the vertices they connect constitute a Nim-heap, of which only adjacent (bold) edges are shown, to avoid cluttering the drawing. Thin lines denote ordinary edges.

All the horizontal lines are thin, and each vertex u_i on this horizontal line connects via a vertical thin edge to a Nim-heap G_i pointing downwards, of size $\lfloor (4i+8)/3 \rfloor$, $i \in \mathbb{Z}^0$. From u_i there also emanates a Nim-heap H_j of size $j = \lfloor (i+2)/3 \rfloor$ pointing upward. Each G_i has a back edge to its top vertex forming a cycle of length $i+1$. Thus G_0 has a loop at the top of its Nim-heap (of size 2). There is an additional back edge to the vertex u_i on the horizontal line, forming a cycle of length $i+3$.

From any vertex u on the horizontal line there is a longest path, via G_i, of length $\lfloor (4i+11)/3 \rfloor$, and the other vertices have shorter maximal length. Thus G is locally path-bounded. But it is not path-bounded, since i can be arbitrarily large.

Sample Problem. Compute an optimal strategy for the 5-token game placed on the 5 starred vertices of G.

To solve this problem, we introduce the generalized Nim-sum ([14], [11], [6]). For any nonnegative integer h we write $h = \sum_{i \ge 0} h^i 2^i$ for the binary encoding of h ($h^i \in \{0,1\}$). If a and b are nonnegative integers, then their *Nim-sum* $a \oplus b = c$, also called *exclusive or, XOR*, or *addition over* GF(2), is defined by $c^i \equiv a^i + b^i \pmod 2$, $c^i \in \{0,1\} \quad (i \ge 0)$.

The Generalized Nim-sum of a nonnegative integer a and $\infty(L)$, for any finite subset $L \subset \mathbb{Z}^0$, is defined by $a \oplus \infty(L) = \infty(L) \oplus a = \infty(L \oplus a)$, where $L \oplus a = \{\ell \oplus a : \ell \in L\}$. The Generalized Nim-sum of $\infty(L_1)$ and $\infty(L_2)$, for any finite subsets L_1, L_2 of $\mathbb{Z}^0$, is defined by $\infty(L_1) \oplus \infty(L_2) = \infty(L_2) \oplus \infty(L_1) = \infty(\emptyset)$. Clearly the Generalized Nim-sum is associative and $a \oplus a = 0$ for every a.

Given any finite or infinite game Γ, we say informally that a *P-position* is any position u from which the *P*revious player can force a win, that is, the opponent of the player moving from u. An *N-position* is any position v from which the *N*ext player can force a win, that is, the player who moves from v. A D-position is any position u from which neither player can force a win, but has a nonlosing next move. The set of all P-, N- and D-positions is denoted by $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{D}$ respectively.

For any finite multiset $\boldsymbol{u} = (u_1, \ldots, u_n)$ of vertices of G on which tokens reside, one token on each u_i, we then have the result [6]:

Proposition. *The P-, N- and D-labels of $\boldsymbol{u} = (u_1, \ldots, u_n)$ in any locally path-bounded digraph G are given by*

$$\begin{aligned}
\mathcal{P} &= \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) = 0\}, \quad \mathcal{D} = \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) = \infty(K),\ \ 0 \notin K\} \\
\mathcal{N} &= \{\boldsymbol{u} \in V : 0 < \sigma(\boldsymbol{u}) < \infty\} \cup \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) = \infty(K),\ \ 0 \in K\}\ . \quad \blacksquare
\end{aligned}$$

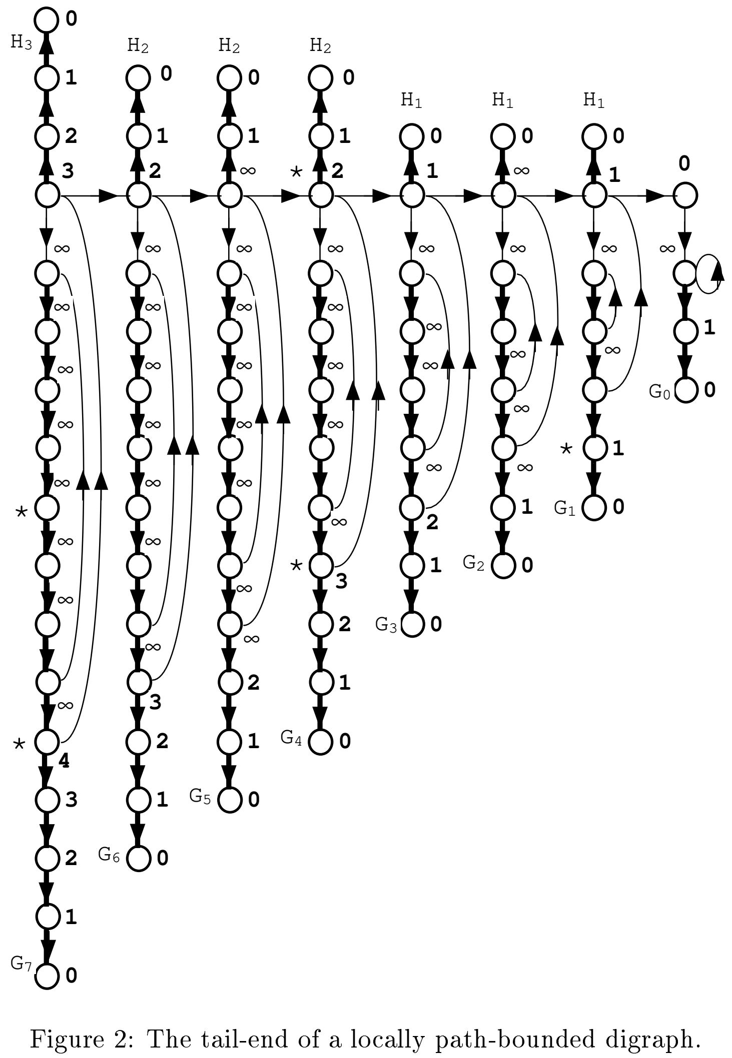

Figure 2: The tail-end of a locally path-bounded digraph.

We are now ready to solve the above problem, by observing that the symbols appearing on Fig. 2 are the γ-values of G. Simply check that they satisfy the conditions of Definition 2. In particular, **B** of Definition 2 is satisfied if every vertex on the horizontal line with γ-value $< \infty$ gets a counter-value between ω and $\omega 2$, and every vertex in the Nim-heaps with γ-value $< \infty$ is assigned a counter value $< \omega$, which is clearly feasible.

For the 5 starred vertices we then have $1 \oplus 3 \oplus 2 \oplus 4 \oplus \infty(0,1,2,3,4) = 4 \oplus \infty(0,1,2,3,4) = \infty(4,5,6,7,0)$, which contains 0, hence the position is in $\mathcal{N}$. Thus the player moving from this position can win by going to a position of Nim-sum 0, namely, pushing the token on the infinity label to 4. Indeed the resulting Nim-sum is $1 \oplus 3 \oplus 2 \oplus 4 \oplus 4 = 0$.

We remark that any tokens on two vertices with γ-value ∞ is a draw position, no matter where the other tokens are, if any. Also note that for realizing a win in this game we do not really need a counter function.

Epilogue

We have defined locally path-bounded digraphs, and shown that the generalized Sprague-Grundy function γ exists on such digraphs with finite, though not necessarily bounded, values. Of course local path-boundedness is only a sufficient condition for the existence of γ. Any finite or infinite digraph without a leaf, satisfies trivially $\gamma(u) = \infty$ for all its vertices u.

A large part of combinatorial game theory is concerned, however, with digraphs which do have leaves. If we exclude digraphs without leaves, then the condition of local boundedness is, in a sense, best possible, as stated in the last part of Theorem 1.

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