Why are Games Exciting and Stimulating? *

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Teach youth to suit their inclinations — Proverbs 22, 6.

Abstract

A taste of the lure of games. It serves a dual purpose: (i) an introduction for the uninitiated, and (ii) a way to introduce youth and others to mathematics in a pleasing and delightful way.

1 The lure of games

Games have a natural appeal, that entices both amateurs and professionals to become addicted to the subject. What is the essence of this appeal? Perhaps the urge to play games is rooted in our primal beastly instincts; the desire to corner, torture, or at least dominate our peers. A common expression of these dark desires is found in the passions roused by local, national and international tournaments. An intellectually refined version, well hidden beneath the façade of scientific research, is the consuming drive "to beat them all", to be more clever than the most clever, in short — to create the tools to *Math-mas*ter them all in hot *comb*inatorial *comb*at! Reaching this goal is particularly satisfying and sweet in the context of combinatorial games, in view of their inherent high complexity.[1]

*Invited presentation for the traveling games exhibition GAMES & SCIENCE, SCIENCE & GAMES, which began in Göttingen, Germany, July 17 – Aug 21, 2005. The German translation, by Niek Neuwahl, is displayed at the exhibition in posters. Readers interested in mathematical background, too technical to be included here, are invited to contact the author. Thanks to Roger Eggleton for helpful editorial remarks.

[1]For application-oriented readers: in addition to a natural appeal of the subject, there are applications or connections to various areas, including complexity, logic, graph and matroid theory, networks, error-correcting codes, surreal numbers, on-line algorithms, biology and — analysis and design of mathematical and commercial games!

The mainstream of games consists of two-player games with perfect information (unlike some card games where information is hidden), without chance moves (no dice), and *outcome* restricted to (lose, win), (tie, tie) and (draw, draw) for the two players who move alternately (no passing). A tie is an end position with no winner and no loser, as may occur in tic-tac-toe for example. A draw is a "dynamic tie", i.e., a non-end position such that neither player can force a win, but each can find a next non-losing move.

2 Nim

The simplest of games is *Nim*: Place identical marbles on a directed graph, such as depicted in Fig. 1, say on circles 1, 2 and 4. A move consists of selecting

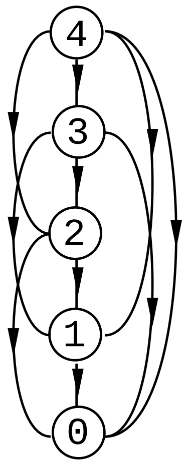

Figure 1: The game of Nim.

a marble and moving it to a neighboring circle, in the direction of an arrow. Double occupancy is permitted. The player first unable to move loses, and the opponent wins. Before reading on, answer the following

Question 1. Can you win from the given position? If so, by what move?

Nim has a very simple winning strategy. Write the number of each occupied circle in binary notation, then add them without carry, such as a kindergarten child might do. If the *Nim-sum* is zero you cannot win by beginning to play from that position: *every* move necessarily makes the Nim-sum nonzero. But if it's nonzero, there is always a move that makes it 0, which is also the value at the end of the game when all the marbles are in position 0, which is therefore a winning move. The Nim-sum of the given position is displayed in Fig. 2(a). The move $4 \to 3$ causes the Nim-sum to become 0. It is a winning move – in this case, the only winning move.

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        4  100          3  011
        2  010          2  010
        1  001  ---->   1  001
           ---             ---
Nim-sum    111             000

          (a)             (b)
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Figure 2: Two Nim-sums in a play of Nim.

3 Fundamental question

Nim and chess and go belong to the same family of games. Why does the former have such a simple strategy, whereas chess and go seem to be so complex?

Well, there are several mathematical differences between these games. Here are a few of them.

• **Cycles**. Nim-like games are finite and "acyclic": no position is assumed twice, but this is not true of chess-like games.

• **Marble interactions**. In Nim-like games, marbles coexist peacefully, even on the same circle, whereas they interact in various ways such as jumping, deflecting, capturing, etc., in chess-like games.

• **Partisanship.** A game is *impartial* if the set of (direct) followers of every position is the same for the two players. If this doesn't hold for all positions, the game is *partisan*. Nim-like games are impartial, whereas chess-like games are partisan (the "black" player cannot move a white piece and vice versa).

In order to throw some light on the fundamental question let us adopt the Roman Cæsars' motto:

DIVIDE AND CONQUER .

Thus, instead of trying in vain to scale the vertical cliff separating Nim from chess, we choose to ascend along scenic trails that lead from sea-level Nim to alpine chess at a moderate gradient, via intermediate games sampled from various strategic viewpoints along the trails. This is done by attacking the above-mentioned mathematical differences between Nim and chess individually, rather than facing them simultaneously.

4 Cycles

Fig. 3 depicts a directed graph, played the same way as Nim, but it has cycles. For example, place identical marbles on circles A, B, E, F. A move consists of selecting a marble and moving it to a neighboring circle along a directed arrow. Multiple occupancy is permitted. The player first unable to move loses and the opponent wins. If there is no last move, the outcome is a draw.

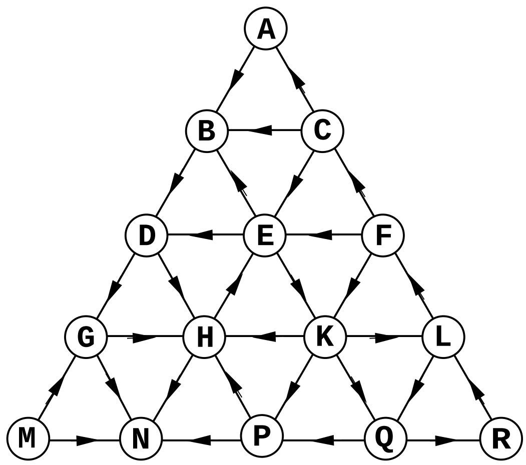

Figure 3: Going in circles.

Question 2. From the given position, can the first player win, or at least draw, or can the opponent win?

Notice that draws are indeed possible in this game. For example, place a single marble on Q. The first player will not go to P, since then the opponent can go to N, winning. Therefore the first player moves to R. It is not difficult to see that as long as the players stick to the region F, K, L, Q, R, they maintain a draw, but a player venturing out of this sanctuary can be defeated by the opponent.

5 Marble interactions

In Fig. 4, identical marbles are placed, say on the starred circles. A move consists of selecting a marble and moving it to a neighboring circle along a directed arrow. If the latter is occupied, *both* marbles are *annihilated* and removed from the game. Once again, the first player unable to move loses, and the opponent wins. Notice that the 3 upper graphs are identical, and so are the two lower ones. The only difference between an upper and a lower graph is the orientation of the top arrows. So things look easy, hence the name "Innocent Marbles".

Question 3. From the given position, can the first player win, or at least draw, or can the opponent win?

Actually, this may not be too easy to answer for the uninitiated, so let's begin with two easier questions.

Question 4. Place 4 marbles: one on z_0 and one on z_2 in one of the graphs; and one on z_0 and one on z_2 in another copy. We ask the same question as in

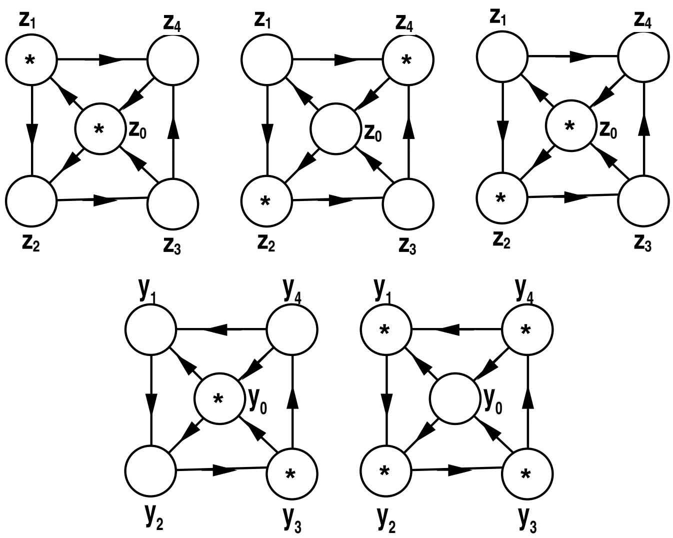

Figure 4: A game of Innocent Marbles. ©1978-2005 A.S. Fraenkel

Question 3 for this position.

Question 5. Place 4 marbles: one on z_0 and one on z_2 in one of the graphs; and one on y_0 and one on y_2 in another copy. We ask the same question as in Question 3 for this position.

Now that you have mastered Innocent Marbles, you are ready to attack the solar system as a homework problem. Consider the stellar configuration marked by letters in "Interstellar encounter with Jupiter" (Figure 5), where J is Jupiter, the other letters are various fragments of the Shoemaker–Levy comet, and all the vertices are "space-stations". A move consists of selecting Jupiter or a fragment, and moving it to a neighboring space-station along a directed trajectory. Any two bodies colliding on a space-station explode and vanish in a cloud of interstellar dust. Note that 6 space-stations are "black holes", where a body is absorbed and cannot escape. Both players are viciously bent on making the final move to destroy this solar subsystem. Is the given position a win for player I or for player II? Or is it a draw, so that a part of this subsystem will exist forever? And if so, can it be arranged for Jupiter to survive as well? (An encounter of the Shoemaker–Levy comet with Jupiter took place in mid-July, 1994.)

Note. If an odd number of marbles is placed on any of the 5 graphs of Innocent Marbles, the outcome of play is trivially a draw, since marbles can disappear only in pairs. But in Fig. 5, the placement of any number of marbles makes a priori sense, because of the black holes.

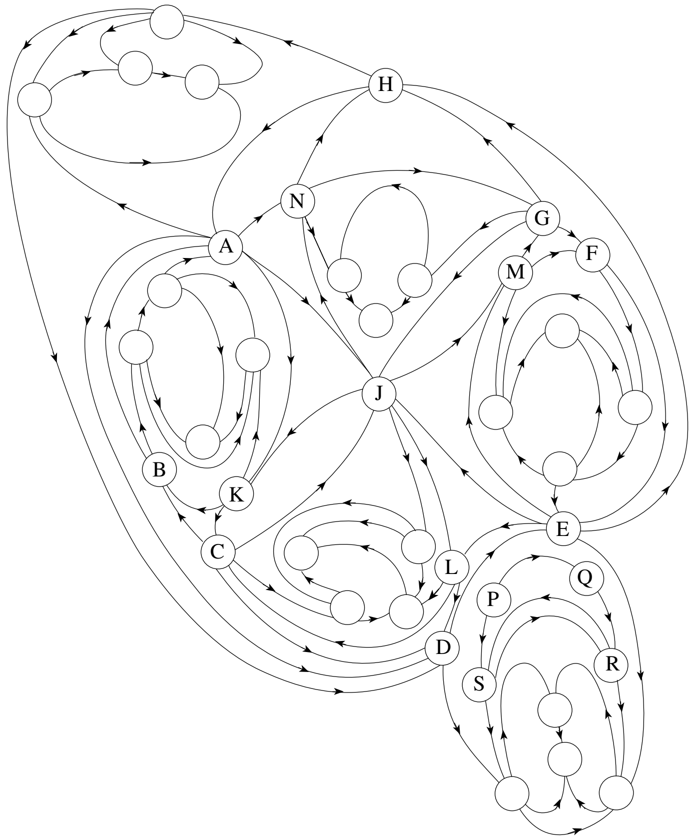

Figure 5: Interstellar encounter with Jupiter. ©1978-2005 A.S. Fraenkel

6 Partisanship

Two players, Vera and Howie, tile a 10×11 "chessboard" by alternately covering a pair of adjacent squares with a domino (Fig. 6). Vera tiles vertically and Howie tiles horizontally. The player first unable to move loses and the opponent wins. Who can win? Note that draws are not possible. On a 2×1 board, Vera wins, on a 1×2, Howie wins, and on a 2×2, the first player wins.

Question 6. Determine the winner for all $m \times n$ boards, with m at most 3 and n at most 4.

7 Partisanship with marble interaction

The game board for the game Arrows, which is featured in this exhibition, is shown in Fig. 7. A team of 7 glass marbles is placed on bases A, B, C, D, E, F, G

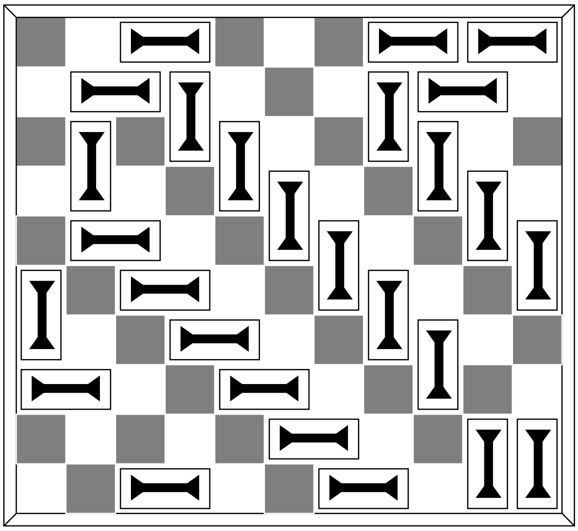

Figure 6: Domineering position after the 14th move of Vera.

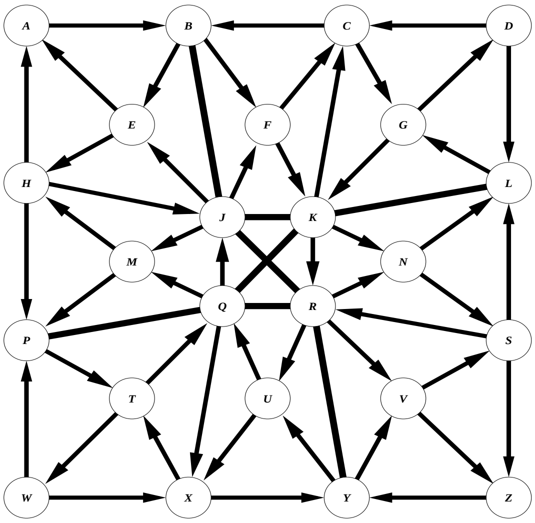

Figure 7: Arrows. ©1975-2005 R.B. Eggleton, A.S. Fraenkel

and a team of 7 steel marbles on T, U, V, W, X, Y, Z. Gladys can move only glass marbles, Steve only steel marbles. At each turn one marble is moved in a straight line from one base to a neighboring base in the direction of the arrow if the two bases are joined by an arrow, or along the line (in either direction) if the two bases are joined by a line. If a player lands on a base occupied by an opposing marble, the latter marble is captured and removed from the board. A marble cannot be moved onto a base that is already occupied by a marble of the same team. The players move alternately. The winner is the player who captures the whole team of 7 opposing marbles.

Question 7. Can the first player in Arrows win or at least draw?

8 Mathgames and playgames

Now is the time to answer some of the earlier questions.

The answer to Question 2 is that there is a unique winning move, namely to move $F \to C$. Every other move leads to a draw or to defeat. For example, see Conway's *On Numbers and Games*[2], where this graph is credited to the author. Conway asked how long it would take Garry Kasparov to find this unique move.

The answer to Question 3 is that there is a unique winning move, which is $y_0 \to y_2$. As for Question 4, it should be clear that the second player can at least draw, by mimicking in one copy the move made by the first player in the other. But it can be shown that the second player can actually win. Regarding Question 5, the outcome is a draw, which can be maintained by moving around on the perimeter of the y copy. These results can be gleaned from the author's "Complexity of games".[3]

These games have the following common features:

(i) They have an "easy" and "fast" strategy.[4]

(ii) They leave the uninitiated layman perplexed, because they lack a *boardfeel*: the loser doesn't know whether his position is weak or strong, and may not even sense that defeat is imminent, only 1 or 2 steps away. The games do not fire the imagination of laymen, though Innocent Marbles may be a little more interesting than the game in Fig. 2.

(iii) They are challenging to mathematicians, who will begin searching for an efficient strategy.

(iv) They become trivial to play, at least to mathematicians, once the underlying theory and strategy are known.

We dub games with these features *mathgames*.

As for Domineering (Fig. 6), several computations, beginning with Conway's book mentioned above, have determined the winner for small boards, but no general method for deciding the winner on arbitrary boards is known. I don't know the answer to Question 7, but:

[2] J.H. Conway *On Numbers and Games*, Ch. 11 (A K Peters, 2001)

[3] R.K. Guy (ed.), *Combinatorial Games*, Proc. Sympos. Appl. Math. Vol. 43, pp. 111–153 (Amer. Math. Soc., Providence, RI, 1991)

[4] Known as "polynomial time strategy".

(i) The game Arrows is "hard".[5] For such games, it is unlikely that a precise strategy can be computed in reasonable time.

(ii) The game Arrows has a boardfeel and is therefore interesting to play. It's a *playgame*, one of the easiest "hard" games!

(iii) The board was designed so that no player can win by imitating the opponent's moves. Moreover, no position consisting of a single glass token and a single steel token is a draw; one or the other can always force a win.

9 View from the summit

Ascending from Nim, we have reached the alpine heights of chess, which is also "hard".[6] What magnificent vistas can be viewed from the summit?

Low-lying Nim-like games are mathgames that have considerable mathematical interest. Further, they have perfect strategies, but players unaware of the strategy remain dumbfounded. The games have no boardfeel.

High-altitude games, especially those that are partisan and have marble interactions, are hard; exact strategies are difficult to achieve, but they usually have a boardfeel, and are candidates to becoming playgames. Elwyn Berlekamp and his students have analyzed a number of endgames of playgames, notably the game of Go.[7]

Thus, paradoxically, games whose strategy we know perfectly are those most of us don't know to play, and none of us enjoys to play, whereas hard games for which we don't have a perfect strategy, portray a challenging boardfeel which enables even the nonexpert player to sense whether he is in a weak or strong position.

10 Concluding remarks

Whereas in everyday life we strive towards solutions or at least approximate solutions which are "easy", there are two important human activities in which high complexity is appreciated. These are cryptography (covert warfare) and games (overt warfare). The desirability of high complexity in cryptography — at least for the encryptor! — is clear. As we indicated above, it is also desirable for games. Incidentally, it's no accident that games and cryptography team up: in both there are adversaries, who pit their wits against each other. But games, where "universal quantifiers" abound, are much harder than cryptography, which is an "existential" problem. This makes them the more challenging and fascinating of the two, besides being fun!

[5] There is a precise mathematical definition for "hardness", called "NP-hardness", which is an "asymptotic" complexity too technical to elaborate on here.

[6] A.S. Fraenkel and D. Lichtenstein, "Computing a perfect strategy for $n \times n$ chess requires time exponential in n", *J. Combinatorial Theory* (Ser. A) **31** (1981) 199–214.

[7] E. Berlekamp and D. Wolfe, *Mathematical Go*, A K Peters, 1994.