On the recurrence $f_{m+1} = b_m f_m - f_{m-1}$ and applications

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There's a proposition proclaimed to be new;
it existed in eternity before us.
The early authors are forgotten;
and also the contemporary ones won't have any remembrance,
though they are modern.

(*Ecclesiastes* 1, 10 − 11, free translation from Hebrew.)

Abstract

We point out that a natural and direct way of analyzing the recurrence $f_{m+1} = b_m f_m - f_{m-1}$ is by means of continued fractions theory. Applications to exotic numeration systems and to games are given, and previous examinations of the special case $f_{m+1} = 6f_m - f_{m-1}$ are pointed out.

Denote by $\mathbb{Z}$, $\mathbb{Z}^0$ and $\mathbb{Z}^+$ the set of integers, nonnegative integers and positive integers respectively. If $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{Z}^+$ $(i \ge 1)$, then

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} := [a_0, a_1, a_2, \ldots] \ , \tag{1}$$

is a *simple continued fraction.* Its *convergents* are: $p_n/q_n = [a_0, \ldots, a_n]$ $(n \ge 0)$. Putting $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, $q_0 = 1$, we then have $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ $(n \ge 1)$. See e.g., [10, Ch. 10], [12].

Let u_n stand for either p_n or q_n, with the understanding that in each formula, either all u_i denote p_i or all denote q_i.

Proposition 1. *Let α be given by (1). Suppose that for some $m \geq 2$, $a_m = a_{m+2} := a$. Then $u_{m+2} = (aa_{m+1} + 2)u_m - u_{m-2}$.*

Proof. We have $u_{m+2} = au_{m+1} + u_m = a(a_{m+1}u_m + u_{m-1}) + u_m$. Since $au_{m-1} = u_m - u_{m-2}$, the result follows. ∎

The recurrence of Proposition 1 is essentially that of the title.

Throughout we put $a_0 = 1$ in (1). We consider the special case of (1) when $a_{2m} = a$ for all $m \in \mathbb{Z}^+$. Put

$$\varepsilon = \begin{cases} 0 & \text{if } u_i = q_i \\ 1 & \text{if } u_i = p_i \end{cases} \qquad (i \geq 1).$$

Letting $u_{-2} = 1 - \varepsilon a$, $u_0 = 1$, Proposition 1 then gives

$$u_{2m} = (aa_{2m-1} + 2)u_{2m-2} - u_{2m-4} \quad (m \geq 1).$$

If we now put $aa_{2m+1} + 2 = b_{2m}$, halve all the indices and write f instead of u, we get precisely the title's recurrence, namely,

$$f_m = b_m f_{m-1} - f_{m-2}. \tag{2}$$

This case was considered in [6, §4], where also the following was proved.

Proposition 2. *Put $a_{2m} = a$ in (1) for all $m \in \mathbb{Z}^+$. Then every positive integer has precisely one representation of the form $N = \sum_{i=0}^{n} d_{2i}u_{2i}$, where the digits d_{2i} are nonnegative integers satisfying*

$$0 \leq d_{2i} \leq aa_{2i+1} + 1 \quad (i \geq 1), \qquad 0 \leq d_0 \leq a(a_1 + \varepsilon)$$

and the additional condition: if for some $0 \leq k < l \leq n$ d_{2l} and d_{2k} attain their maximal values, then there exists j satisfying $k < j < l$ (so actually $l - k \geq 2$) such that $d_{2j} < aa_{2j+1}$.

The immediate motivation for recounting the preceding facts is [2], where the recurrence

$$f_n = 6f_{n-1} - f_{n-2}, \tag{3}$$

a special case of (2), was considered *inter alia*. It was followed by the comment, "This recurrence cries out for a combinatorial interpretation. Finding this interpretation is an open question." In [1] an application of (3) to regular languages was given. Irrespective of any applications, we suggest that the theory of continued fractions is the "right" tool for viewing recurrences of this form, because it leads naturally and directly to them. Applications also lie close by, in view of the immediate use for exotic numeration systems, which, among other things, lead to winning strategies for certain combinatorial games. Such numeration systems can also be used for efficient compression of sparse vectors — see [9].

For $b_n = aa_{2n+1} + 2 = 6$, i.e., $aa_{2n-1} = 4$ for all $n \in \mathbb{Z}^0$, we get four solutions to the recurrence (3), namely the p_{2i}, q_{2i} sequences of the simple continued fractions

$$[1, (4,1)^\infty] = (\sqrt{2}+1)/2, \quad [1, (1,4)^\infty] = 2\sqrt{2} - 1, \quad [1, 2^\infty] = \sqrt{2}. \tag{4}$$

They can be specified by their initial conditions $p_2 = 6$, $p_2 = 9$, $p_2 = 7$, respectively, where $p_0 = q_0 = 1$, $q_2 = 5$ throughout. The third p_{2i} sequence, namely

$$\{p_{2i} : i \geq 0\} = \{1, 7, 41, 239, 1393, 8119, 47321, \dots\}, \tag{5}$$

is the solution of (3) considered in both [2] and [1]. Each of the four sequences generates a numeration system satisfying Proposition 2. Of course we can get infinitely many solutions for (3) by putting $aa_{2n-1} = 4$ for all large n, instead for all $n \in \mathbb{Z}^0$. This amounts to assigning different initial values to the recurrence (3).

The simplest case of (2) is when $a = a_{2m+1} = 1$ for all $m \geq 0$. Then (1) becomes the *Fibonacci* situation, where $\alpha = [1^\infty] = (1 + \sqrt{5})/2 := \phi$ is the golden section. The p_{2i} then constitute the "even" Fibonacci numbers $1, 3, 8, 21, 55, \dots$ and the q_{2i} the "odd" ones: $1, 2, 5, 13, 34, \dots$ with $p_{2i}/q_{2i} \to \phi$ as $i \to \infty$. By Proposition 2 the two corresponding numeration systems are ternary, with the additional condition that between any two 2 digits there must be a 0-digit. This numeration system was used in [3] for investigating irregularities of distribution of sequences. A recent application to "combinatorial matrices" and games was given in [7].

We point out that a somewhat more direct way to get the recurrence (2) is to consider *quasiregular* (or *semiregular*— halbregelmässig) continued fractions, which have the form

$$\beta = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots}}} := [b_0; a_1, b_1; a_2, b_2; \dots] \ ,$$

with $b_i \in \mathbb{Z}^+$ and $|a_i| = 1$ for all i. In [12, Ch. 5] it is proved that quasiregular continued fractions converge. Their convergents $p_n/q_n = [b_0; a_1, b_1; \dots; a_n, b_n]$ satisfy $p_{-1} = 1$, $p_0 = b_0$, $q_{-1} = 0$, $q_0 = 1$, $u_n = b_n u_{n-1} + a_n u_{n-2}$, $(n \geq 2)$, where again u_n stands for either p_n or q_n.

Putting $a_n = -1$ for all $n \geq 1$, we see that both p_n and q_n satisfy our recurrence (2). For $a_i = -1$ and $b_i = 3$ for all i we get $\beta = 3 - 1/\beta$, so $\beta = \phi^2 = \phi + 1$, where the p_i $(i \geq -1)$ constitute the "even" Fibonacci numbers, $q_i = p_{i-1}$ for all $i \geq 0$, with $p_i/q_i \to \phi^2$. For $a_i = -1$ and $b_0 = 7$, $b_i = 6$ for all $i \geq 1$, the p_i $(i \geq -1)$ constitute the sequence $1, 7, 41, 239, 1393, 8119, 47321, \dots$ considered above, and the q_i $(i \geq 0)$ form the sequence $1, 6, 35, 204, 1189, 6930, \dots$ with $p_i/q_i \to 4 + 2\sqrt{2}$.

The characteristic equation of (3) is $x^2 - 6x + 1 = 0$, with solutions $\alpha = 3 + 2\sqrt{2} = \beta^2$, $\alpha^{-1} = 3 - 2\sqrt{2} = \beta^{-2}$, where $\beta = \sqrt{2} + 1$, $\beta^{-1} = \sqrt{2} - 1$. This leads to

$$f_m = \frac{(f_1 - f_0\alpha^{-1})\alpha^m + (-f_1 + f_0\alpha)\alpha^{-m}}{4\sqrt{2}} \ .$$

For the three continued fractions (4), the three sequences p_{2i} (with $f_0 = p_0$,

$f_1 = p_2$), and the sequence q_{2i} ($f_0 = q_0$, $f_1 = q_2$) then satisfy, respectively,

$$f_m = \frac{\alpha^{m+1} - \alpha^{-(m+1)}}{4\sqrt{2}} = \frac{\beta^{2m+2} - \beta^{-(2m+2)}}{4\sqrt{2}} \ ,$$

$$f_m = \frac{(3+\sqrt{2})\alpha^m - (3-\sqrt{2})\alpha^{-m}}{2\sqrt{2}} = \frac{(3+\sqrt{2})\beta^{2m} - (3-\sqrt{2})\beta^{-2m}}{2\sqrt{2}} \ ,$$

$$f_m = \frac{\beta^{2m+1} - \beta^{-(2m+1)}}{2} \ , \tag{6}$$

$$f_m = \frac{\beta^{2m+1} + \beta^{-(2m+1)}}{2\sqrt{2}} \ .$$

The solution (6) appears in [1] and is credited there to [11].

As a sample-application, we shall now show that the sequence (5) can be used to formulate a polynomial strategy for a *generalized Wythoff* game. We shall only state and illustrate the facts, which are special cases of theorems proved in [5].

A generalized Wythoff game is defined by means of a positive integer parameter t. Given two piles of tokens, two players move alternately. The moves are of two types: remove any positive number of tokens from a *single* pile, or take from both piles, say k (> 0) from one and ℓ (> 0) from the other, provided that $|k - \ell| < t$. The player first unable to move loses, and the opponent wins. For $t = 1$, the second-type move permits to remove only the *same* number of tokens from both piles. This is the classical Wythoff game [13], [4], [14]. Another generalization appears in [8].

We denote game positions by (x, y) with $x \le y$, and proceed to examine the game for $t = 2$. Clearly $(0, 0)$ is a player II (second player) winning position, since player I (the first player) cannot move, so loses. Any player II winning position will be called a P-position (*P*revious player winning position), and the set of all P-positions is denoted by $\mathcal{P}$. Similarly, any player I winning position is called an N-position (*N*ext player win), and the set of all N-positions is $\mathcal{N}$.

The position $(1, 3)$ is also in $\mathcal{P}$, since it is verified easily that player I cannot move to 0, but player II can do so after any move of player I. The P-positions (A_n, B_n) for $n \in \{0, \ldots, 10\}$ are listed in Table 1, which has a rather interesting general structure, stated in the next proposition. Before reading on, try to guess the values (A_{11}, B_{11}) and (A_{12}, B_{12}).

Let S denote any finite subset of $\mathbb{Z}^0$ and $\overline{S} = \mathbb{Z}^0 \setminus S$. We define the Minimum EXcluded value of S as $\operatorname{mex} S = \min \overline{S} =$ smallest element of $\mathbb{Z}^0$ not in S. For any nonnegative integer n, let

$$A_n = \operatorname{mex}\{A_i, B_i : 0 \le i < n\}, \quad B_n = A_n + tn \quad (n \in \mathbb{Z}^0). \tag{7}$$

Note that $A_0 = B_0 = 0$. The following holds.

Proposition 3. *$\mathcal{P} = \bigcup_{i=0}^{\infty} \{(A_i, B_i)\}$, where A_n and B_n are given by* (7).

Table 1: THE FIRST FEW P-POSITIONS OF THE GENERALIZED WYTHOFF GAME FOR $t = 2$.

n	A_n	B_n
0	0	0
1	1	3
2	2	6
3	4	10
4	5	13
5	7	17
6	8	20
7	9	23
8	11	27
9	12	30
10	14	34

The mex property implies that if we put $A = \cup_{n=1}^{\infty} A_n$, $B = \cup_{n=1}^{\infty} B_n$, then A, B are *complementary*, i.e., $A \cup B = \mathbb{Z}^+$, $A \cap B = \emptyset$. Hence Proposition 3 permits to give a winning strategy for any given game position (x, y) with $x \le y$. Table 1 is computed up to the smallest integer $A_n \ge x$. If there is equality and also $y = B_n$, then $(x, y) \in \mathcal{P}$. Otherwise $(x, y) \in \mathcal{N}$, and the proof of Proposition 3, which is constructive, shows how to win by moving to a P-position.

Let's examine the complexity of this strategy. Table 1 has to be computed only up to the smallest integer $A_n \ge x$. It is not hard to see that $n = \Theta(x)$, so the strategy is linear in x. Is this an efficient strategy? Well, the game is *succinct*, i.e., the input size is $\Theta(\log x + \log y)$, so the strategy is actually exponential, also called *intractable* in computational complexity theory (unless y is exponentially larger than x).

Is there a polynomial strategy? The next proposition implies a positive answer to this question.

Proposition 4. *$A_n = \lfloor n\sqrt{2} \rfloor$, $B_n = \lfloor n\sqrt{2} \rfloor + 2n$ for all $n \in \mathbb{Z}^0$, where A_n and B_n are defined in* (7).

(Check that the first few values of Table 1 satisfy Proposition 4.) Let (x, y) with $x \le y$ be a given game position. Proposition 4 implies that either $x = \lfloor n\sqrt{2} \rfloor = A_n$ where $n = \lfloor (x+1)/\sqrt{2} \rfloor$, or else, by complementarity, $x = \lfloor n(\sqrt{2}+2) \rfloor = B_n$, where $n = \lfloor (x+1)/(\sqrt{2}+2) \rfloor$. These facts suffice to provide a polynomial winning strategy, since it suffices to compute $\sqrt{2}$ to a precision of $O(\log x)$ digits, and store it in $O(\log x)$ words.

For formulating yet another polynomial strategy, we use the sequence $\{p_i\}$ of the numerators of the convergents of the simple continued fraction expansion of $\sqrt{2}$, namely, $\{p_i : i \ge 0\} = \{1, 3, 7, 17, \dots\}$ as basis elements of a special ternary numeration system.

Proposition 5. *Every nonnegative integer N has precisely one represen-*

Table 2: An exotic ternary representation of the first few positive integers.

17	7	3	1	n
			1	1
			2	2
		1	0	3
		1	1	4
		1	2	5
		2	0	6
	1	0	0	7
	1	0	1	8
	1	0	2	9
	1	1	0	10
	1	1	1	11
	1	1	2	12
	1	2	0	13
	2	0	0	14
	2	0	1	15
	2	0	2	16
1	0	0	0	17
1	0	0	1	18
1	0	0	2	19
1	0	1	0	20

tation of the form

$$N = \sum_{i \geq 0} d_i p_i, \quad 0 \leq d_i \leq 2; \qquad d_{i+1} = 2 \Rightarrow d_i = 0 \qquad (i \in \mathbb{Z}^0).$$

The representation of the first few positive integers in this numeration system is depicted in Table 2. Is there any connection with Table 1? Well, note that the pair $(1,3)$ has representation $(1,10)$ where the first component ends in an even number (zero) of 0s, and the second component is the "left shift" of 1: The left shift of $N = \sum_{i \geq 0} d_i p_i$ is $L(N) = \sum_{i \geq 0} d_i p_{i+1}$. These observations hold in general:

Proposition 6. *Let $N \in \mathbb{Z}^+$. Then $N \in A_n$ if and only if the smallest nonzero basis element of the representation of N belongs to the sequence* (5). *Moreover, if $N \in A_n$, then $B_n = L(A_n)$.*

Given any position (x, y) with $x \leq y$ of our game, we represent x in the special ternary system and observe whether it ends in an even number of 0s (then the smallest nonzero basis element belongs to the sequence (5)) or not. This can be done in polynomial time. Thus Proposition 6 constitutes another application of the sequence (5) to the construction of a polynomial strategy for our game.

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